UKF滤波

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卡尔曼滤波与组合导航原理(4-6讲)

卡尔曼滤波与组合导航原理

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14 UKF滤波(Unscented KF, 无迹/无味卡尔曼滤波)

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His principal research interests include distributed data fusion, vehicle localization and tracking, and user interfaces for mobile augmented reality systems.

- Julier S J, Uhlmann J K, Durrant-Whyte H F. A new approach for filtering nonlinear systems[C]//Proc. Am. Contr. Conf.. Seattle,WA, 1995: 1628-1632.
- Julier S J, Uhlmann J K. Unscented filtering and nonlinear estimation[J]. Proc. of the IEEE Aerospace and Electronic Systems, 2004, 92(3): 401-422.

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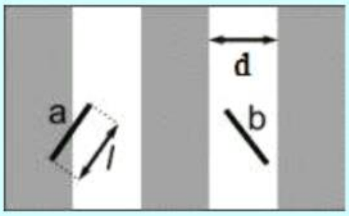
14.1 蒙特卡洛仿真

蒲丰投针问题（1777年,计算π最稀奇的方法）

1）取一张白纸，在上面画上许多条间距为*d*的平行线；

2） 取一根长度为 *l*（*l*<*d*） 的针，随机地向画有平行直线的纸上掷 *n* 次，观察针与直线相交的次数，记为 *m*；

3）计算针与直线相交的概率：$p = \frac{2l}{\pi d} \approx \frac{m}{n}$



实验者	年份	投计次数	π的实验值
沃尔弗(Wolf)	1850	5000	3.1596
斯密思(Smith)	1855	3204	3.1553
福克斯(Fox)	1894	1120	3.1419
拉查里尼(Lazzarini)	1901	3408	3.1415929

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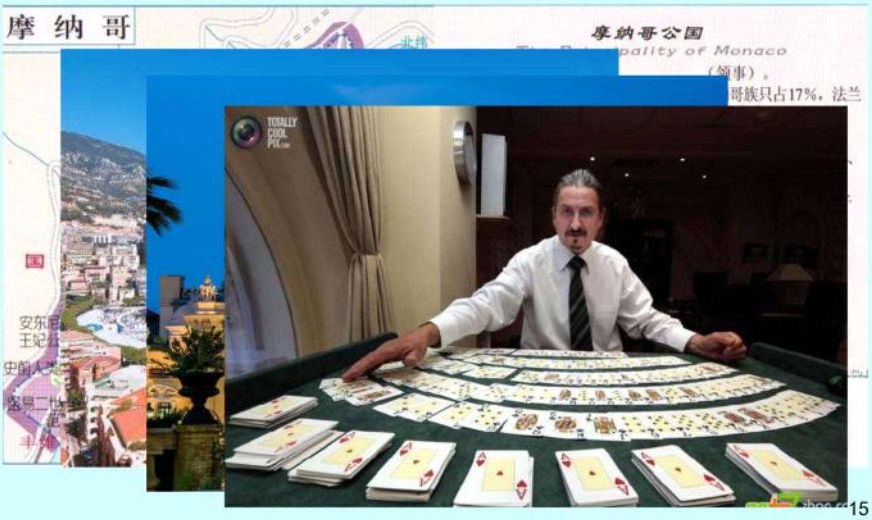
蒙特卡洛仿真名称来源(Monte Carlo)



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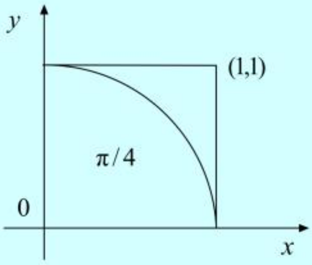
蒙特卡洛仿真举例

【例1】我们都知道，单位圆围成的面积为π，若以圆心为原点建立平面直角坐标系，则它在第一象限内的面积为π/4 ，即有

$$\int_0^1\int_0^{\sqrt{1-y^2}} 1\mathrm{d}x\mathrm{d}y = \pi/4$$



Matlab编写的仿真语句和结果：

```
n = 1000000;
p = unifrnd(0,1,2,n);
PI = 4*sum(sum(p.^2)<1)/n

--> PI = 3.141092
```

收敛精度 $\varepsilon \propto \frac{1}{\sqrt{n}}$ ，收敛速度慢！

仿好的 ε∝10^-3，3位数字精度

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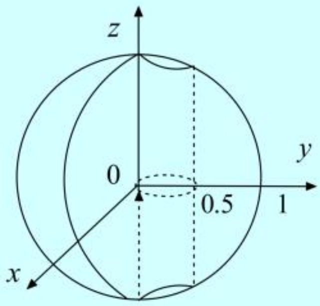
【例2】在单位球体内偏离中心0.25的位置挖去一个半径为0.25的柱体（参见图），试求该球体剩余部分的体积。



Matlab编写的仿真语句和结果：

```
n = 100000;
p = unifrnd(-1,1,3,n);
V = 2^3*sum(sum(p.^2)<1&(p(1,:).^2+
        (p(2,:)-0.25).^2)>0.25^2)/n

--> V = 3.812905
```

特点：解决方法与维数无关，只与边界有关，看边界是否能描述

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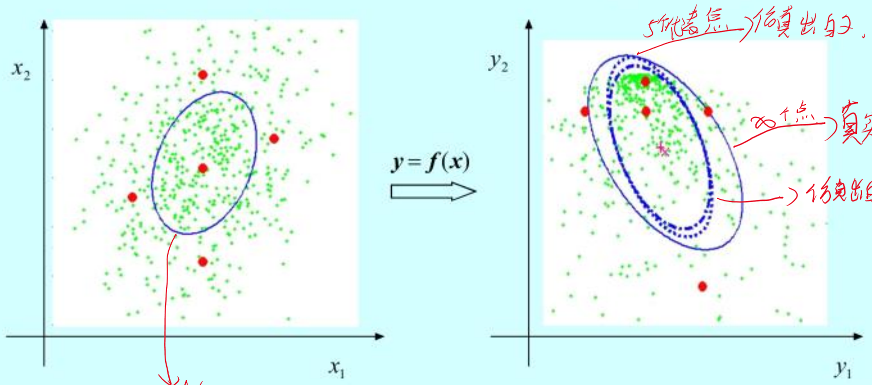
【例3】二维随机向量的非线性变换/概率传播

$$\begin{cases} y_1 = (x_1 - 1)\cdot(x_2 - 0.2) \\ y_2 = -(x_1 - 1)^2 \end{cases} \qquad E[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}] = \begin{bmatrix} 1 & 0.42 \\ 0.42 & 2 \end{bmatrix}$$



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14.2 UT变换(Unscented Transformation)

（1）线性函数的UT变换

假设随机向量的线性变换关系 $\boldsymbol{y} = \boldsymbol{F}\boldsymbol{x}$

$$\begin{cases} \bar{\boldsymbol{x}} = E[\boldsymbol{x}] \\ \boldsymbol{P}_x = E[(\boldsymbol{x}-\bar{\boldsymbol{x}})(\boldsymbol{x}-\bar{\boldsymbol{x}})^{\mathrm{T}}] \end{cases} \Longrightarrow \begin{cases} \bar{\boldsymbol{y}} = \boldsymbol{F}\bar{\boldsymbol{x}} \\ \boldsymbol{P}_y = \boldsymbol{F}\boldsymbol{P}_x\boldsymbol{F}^{\mathrm{T}} \\ \boldsymbol{P}_{xy} = \boldsymbol{P}_x\boldsymbol{F}^{\mathrm{T}} \end{cases}$$

A）矩阵的三角分解 $\boldsymbol{P}_x = \boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$ 　记 $\sqrt{\boldsymbol{P}_x} = \boldsymbol{A}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}\begin{bmatrix} 0 & 3 \end{bmatrix}$$

$$\boldsymbol{P}_x = \sqrt{\boldsymbol{P}_x}\sqrt{\boldsymbol{P}_x}^{\mathrm{T}} = \sum_{i=1}^{n}(\sqrt{\boldsymbol{P}_x})_i(\sqrt{\boldsymbol{P}_x})_i^{\mathrm{T}}$$

B）输入近似成有限个（n个）Sigma点的估计形式

$$\begin{cases} \bar{\boldsymbol{x}} = \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\chi_i \approx \frac{1}{n}\sum_{i=1}^{n}\chi_i \\ \boldsymbol{P}_x = \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}(\chi_i-\bar{\boldsymbol{x}})(\chi_i-\bar{\boldsymbol{x}})^{\mathrm{T}} \approx \frac{1}{n}\sum_{i=1}^{n}(\chi_i-\bar{\boldsymbol{x}})(\chi_i-\bar{\boldsymbol{x}})^{\mathrm{T}} \end{cases}$$

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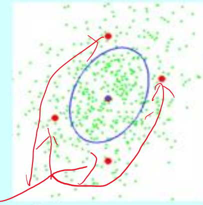
$$P_x = \sqrt{P_x}\sqrt{P_x}^{\mathrm{T}} = \sum_{i=1}^{n}(\sqrt{P_x})_i(\sqrt{P_x})_i^{\mathrm{T}}$$

$$\begin{cases} \bar{x} = \lim\limits_{k\to\infty}\dfrac{1}{k}\sum_{i=1}^{k}\chi_i \approx \dfrac{1}{n}\sum_{i=1}^{n}\chi_i \\ P_x = \lim\limits_{k\to\infty}\dfrac{1}{k}\sum_{i=1}^{k}(\chi_i-\bar{x})(\chi_i-\bar{x})^{\mathrm{T}} \approx \dfrac{1}{n}\sum_{i=1}^{n}(\chi_i-\bar{x})(\chi_i-\bar{x})^{\mathrm{T}} \end{cases}$$



比较，可得 $\frac{1}{\sqrt{n}}(\chi_i-\bar{x}) = \pm(\sqrt{P_x})_i$ 　即 $\chi_i = \bar{x} \pm (\sqrt{nP_x})_i$

C）输入近似成有限个（2n个）Sigma点的估计形式

$$\begin{cases} \bar{x} = \dfrac{1}{2n}\sum_{i=1}^{2n}\chi_i \\ P_x = \dfrac{1}{2n}\sum_{i=1}^{2n}(\chi_i-\bar{x})(\chi_i-\bar{x})^{\mathrm{T}} \end{cases}$$

记Sigma点组成的输入矩阵 $\chi = \left[[\bar{x}]_n + \sqrt{nP_x} \quad [\bar{x}]_n - \sqrt{nP_x}\right]$

得Sigma点组成的输出矩阵 $\eta = \left[[F\bar{x}]_n + F\sqrt{nP_x} \quad [F\bar{x}]_n - F\sqrt{nP_x}\right]$ 　y=Fx

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$$\chi = \left[[\bar{x}]_n + \sqrt{nP_x} \quad [\bar{x}]_n - \sqrt{nP_x}\right]$$
$$\eta = \left[[F\bar{x}]_n + F\sqrt{nP_x} \quad [F\bar{x}]_n - F\sqrt{nP_x}\right]$$

由输入输出Sigma点计算传播统计特征

$$\begin{cases} \bar{y} = \dfrac{1}{2n}\sum_{i=1}^{2n}\eta_i = \dfrac{1}{2n}\sum_{i=1}^{2n}F\bar{x} = F\bar{x} \\ P_y = \dfrac{1}{2n}\sum_{i=1}^{2n}(\eta_i-\bar{y})(\eta_i-\bar{y})^{\mathrm{T}} = \dfrac{1}{2n}\sum_{i=1}^{2n}(F\sqrt{nP_x})(F\sqrt{nP_x})^{\mathrm{T}} = FP_xF^{\mathrm{T}} \\ P_{xy} = \dfrac{1}{2n}\sum_{i=1}^{2n}(\chi_i-\bar{x})(\eta_i-\bar{y})^{\mathrm{T}} = \dfrac{1}{2n}\sum_{i=1}^{2n}(\sqrt{nP_x})(F\sqrt{nP_x})^{\mathrm{T}} = P_xF^{\mathrm{T}} \end{cases}$$

这与理论结果完全一致，精确地捕获了输出的一、二阶矩！ 　线性的思路移植到非线性

（2）非线性函数的UT变换

非线性函数 $y = f(x)$ 　UT变换

$$\begin{cases} \chi = \left[[\bar{x}]_n + \sqrt{nP_x} \quad [\bar{x}]_n - \sqrt{nP_x}\right] \\ \eta_i = f(\chi_i) \\ \bar{y} \approx 1/(2n)\cdot\sum_{i=1}^{2n}\eta_i \\ P_y \approx 1/(2n)\cdot\sum_{i=1}^{2n}(\eta_i-\bar{y})(\eta_i-\bar{y})^{\mathrm{T}} \\ P_{xy} \approx 1/(2n)\cdot\sum_{i=1}^{2n}(\chi_i-\bar{x})(\eta_i-\bar{y})^{\mathrm{T}} \end{cases}$$

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非线性函数UT变换的一般形式

$$\begin{cases}\chi = \left[[\bar{x}]_n + \sqrt{nP_x} \quad [\bar{x}]_n - \sqrt{nP_x}\right] \\ \eta_i = f(\chi_i) \\ \bar{y} \approx 1/(2n)\cdot\sum_{i=1}^{2n}\eta_i \\ P_y \approx 1/(2n)\cdot\sum_{i=1}^{2n}(\eta_i-\bar{y})(\eta_i-\bar{y})^{\mathrm{T}} \\ P_{xy} \approx 1/(2n)\cdot\sum_{i=1}^{2n}(\chi_i-\bar{x})(\eta_i-\bar{y})^{\mathrm{T}}\end{cases} \Rightarrow \begin{cases}\chi = \left[\bar{x} \quad [\bar{x}]_n + \gamma\sqrt{P_x} \quad [\bar{x}]_n - \gamma\sqrt{P_x}\right] \\ \eta_i = f(\chi_i) \\ \bar{y} = \sum_{i=0}^{2n}W_i^m\eta_i \\ P_y = \sum_{i=0}^{2n}W_i^c(\eta_i-\bar{y})(\eta_i-\bar{y})^{\mathrm{T}} \\ P_{xy} = \sum_{i=0}^{2n}W_i^c(\chi_i-\bar{x})(\eta_i-\bar{y})^{\mathrm{T}}\end{cases}$$

其中

$$\begin{cases}\lambda = \alpha^2(n+\kappa)-n \\ \gamma = \sqrt{n+\lambda} \\ W_0^m = \lambda/\gamma^2 \qquad W_0^c = \lambda/\gamma^2 + (1-\alpha^2+\beta) \\ W_i^m = W_i^c = 1/(2\gamma^2) \qquad (i=1,2,\ldots,2n)\end{cases}$$

α 控制Sigma点分布,0~1;

κ 一比例因子,常为0;

$\beta \geq 0$ 另一比例因子,常为2

控制点离均值的远近，非线性越强，λ应越小

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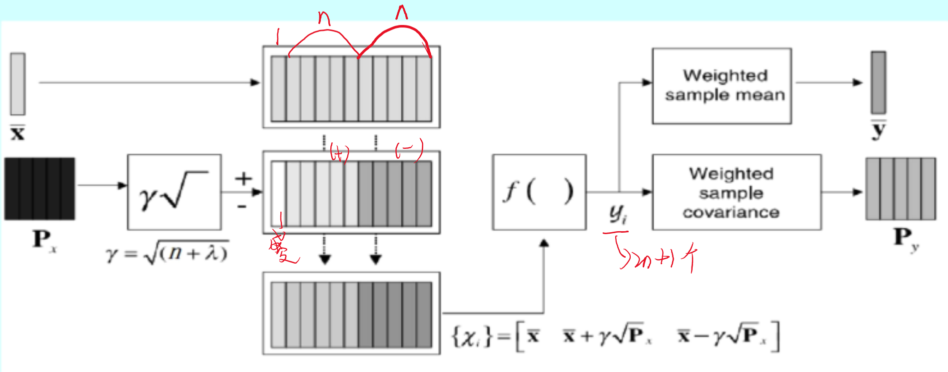
UT变换的直观图示:



非线性函数的一阶矩、二阶矩的概率传播

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14.3 UKF滤波

系统模型

$$\begin{cases} \boldsymbol{x}_k = \boldsymbol{f}(\boldsymbol{x}_{k-1}) + \boldsymbol{B}_{k-1}\boldsymbol{w}_{k-1} \\ \boldsymbol{y}_k = \boldsymbol{h}(\boldsymbol{x}_k) + \boldsymbol{v}_k \end{cases}$$

滤波框架

$$\begin{cases} \boldsymbol{K}_k = \boldsymbol{P}_{xy,k/k-1}\boldsymbol{P}_{y,k/k-1}^{-1} \\ \hat{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_{k/k-1} + \boldsymbol{K}_k(\boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k/k-1}) \\ \boldsymbol{P}_{x,k} = \boldsymbol{P}_{x,k/k-1} - \boldsymbol{K}_k\boldsymbol{P}_{y,k/k-1}\boldsymbol{K}_k^{\mathrm{T}} \end{cases}$$

框架不变
二阶距怎么算变了

状态预测UT变换

$$\begin{cases} \boldsymbol{\chi}_{k-1} = \left[\hat{\boldsymbol{x}}_{k-1} \quad [\hat{\boldsymbol{x}}_{k-1}]_n + \gamma\sqrt{\boldsymbol{P}_{x,k-1}} \quad [\hat{\boldsymbol{x}}_{k-1}]_n - \gamma\sqrt{\boldsymbol{P}_{x,k-1}}\right] \\ \boldsymbol{\chi}_{i,k/k-1}^* = \boldsymbol{f}(\boldsymbol{\chi}_{i,k-1}) \\ \hat{\boldsymbol{x}}_{k/k-1} = \sum_{i=0}^{2n} W_i^m \boldsymbol{\chi}_{i,k/k-1}^* \\ \boldsymbol{P}_{x,k/k-1} = \sum_{i=0}^{2n} W_i^c (\boldsymbol{\chi}_{i,k/k-1}^* - \hat{\boldsymbol{x}}_{k/k-1})(\boldsymbol{\chi}_{i,k/k-1}^* - \hat{\boldsymbol{x}}_{k/k-1})^{\mathrm{T}} + \boldsymbol{B}_{k-1}\boldsymbol{Q}_{k-1}\boldsymbol{B}_{k-1}^{\mathrm{T}} \end{cases}$$

→先用上一时刻的估计值和方差生成sigma点(x)
→均值
→方差

量测预测UT变换

$$\begin{cases} \boldsymbol{\chi}_{k/k-1} = \left[\hat{\boldsymbol{x}}_{k/k-1} \quad [\hat{\boldsymbol{x}}_{k/k-1}]_n + \gamma\sqrt{\boldsymbol{P}_{x,k/k-1}} \quad [\hat{\boldsymbol{x}}_{k/k-1}]_n - \gamma\sqrt{\boldsymbol{P}_{x,k/k-1}}\right] \\ \boldsymbol{\eta}_{i,k/k-1} = \boldsymbol{h}(\boldsymbol{\chi}_{i,k/k-1}) \\ \hat{\boldsymbol{y}}_{k/k-1} = \sum_{i=0}^{2n} W_i^m \boldsymbol{\eta}_{i,k/k-1} \\ \boldsymbol{P}_{y,k/k-1} = \sum_{i=0}^{2n} W_i^c (\boldsymbol{\eta}_{i,k/k-1} - \hat{\boldsymbol{y}}_{k/k-1})(\boldsymbol{\eta}_{i,k/k-1} - \hat{\boldsymbol{y}}_{k/k-1})^{\mathrm{T}} + \boldsymbol{R}_k \\ \boldsymbol{P}_{xy,k/k-1} = \sum_{i=0}^{2n} W_i^c (\boldsymbol{\chi}_{i,k/k-1} - \hat{\boldsymbol{x}}_{k/k-1})(\boldsymbol{\eta}_{i,k/k-1} - \hat{\boldsymbol{y}}_{k/k-1})^{\mathrm{T}} \end{cases}$$

→再生成sigma点(y)

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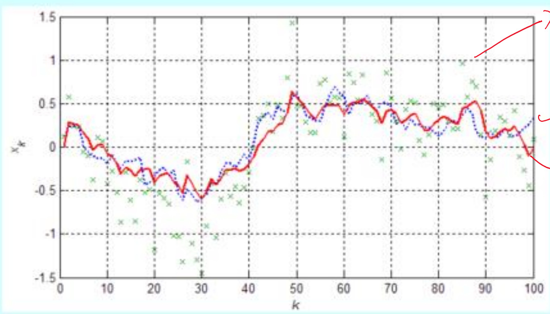
【例1】一维非线性随机系统

$$\begin{cases} x_k = \sin(x_{k-1}) + w_{k-1} \\ y_k = \begin{cases} x_k + v_k & x_k > 0 \\ 2x_k + v_k & x_k \le 0 \end{cases} \end{cases} \qquad w_k \sim WN(0, 0.1^2) \qquad v_k \sim WN(0, 0.3^2)$$

← 此处不可导，EKF就不行了

仿真结果



非线性系统UKF滤波仿真

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【例2】针对非常复杂的非线性系统模型，*近似非线性概率传播比近似非线性函数（泰勒展开线性化）*更简单，此时UKF滤波比EKF滤波更具优势。

非线性方程有输入和输出
就可以用UKF，不必考虑到的
（EKF的展开和Jacobi矩阵

惯导大失准角误差模型

$$\begin{cases} \dot{\boldsymbol{\alpha}} = \boldsymbol{C}_\omega^{-1}\left[(\boldsymbol{I} - \boldsymbol{C}_n^{n'})\boldsymbol{\omega}_{in}^n\right] - \boldsymbol{\varepsilon}^{n'} \\ \delta\dot{\boldsymbol{v}}^n = \left[\boldsymbol{I} - (\boldsymbol{C}_n^{n'})^{\mathrm{T}}\right]\tilde{\boldsymbol{f}}_{sf}^{n'} + (\boldsymbol{C}_n^{n'})^{\mathrm{T}} ?^{n'} \end{cases}$$

其中

$$\boldsymbol{C}_\omega^{-1} = \frac{1}{c\alpha_x}\begin{bmatrix} c\alpha_y c\alpha_x & 0 & s\alpha_y c\alpha_x \\ s\alpha_y s\alpha_x & c\alpha_x & -c\alpha_y s\alpha_x \\ -s\alpha_y & 0 & c\alpha_y \end{bmatrix}$$

c - cos
s - sin

$$\boldsymbol{C}_n^{n'} = \begin{bmatrix} c\alpha_y c\alpha_z - s\alpha_y s\alpha_x s\alpha_z & c\alpha_y s\alpha_z + s\alpha_y s\alpha_x c\alpha_z & -s\alpha_y c\alpha_x \\ -c\alpha_x s\alpha_z & c\alpha_x c\alpha_z & s\alpha_x \\ s\alpha_y c\alpha_z + c\alpha_y s\alpha_x s\alpha_z & s\alpha_y s\alpha_z - c\alpha_y s\alpha_x c\alpha_z & c\alpha_y c\alpha_x \end{bmatrix}$$

摘自：基于欧拉平台误差角的SINS非线性误差模型研究，西北工业大学学报,2009.