# 高斯积分非线性滤波

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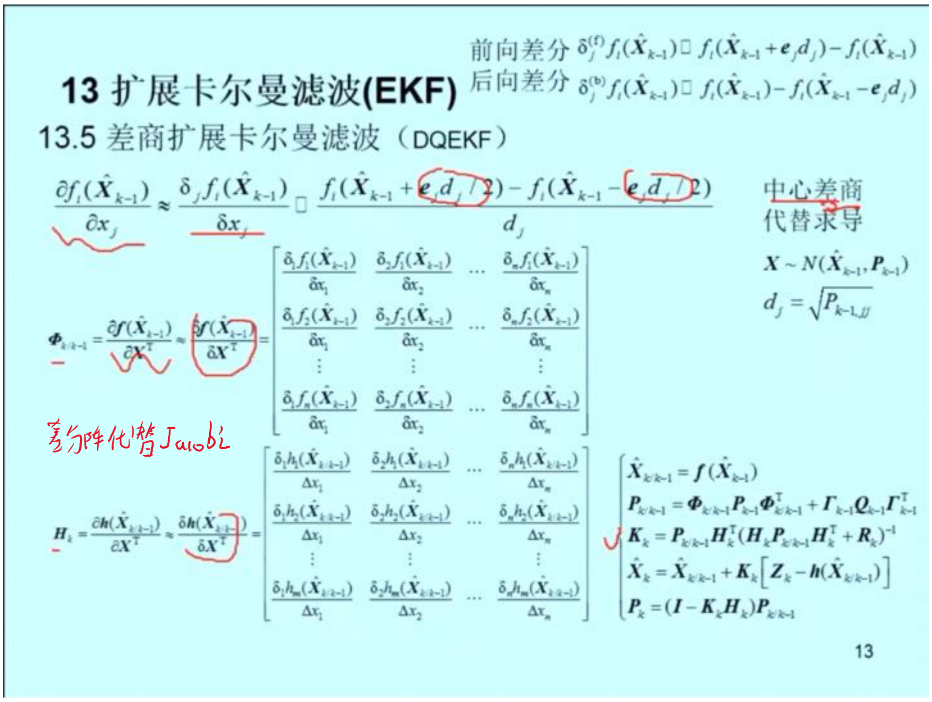
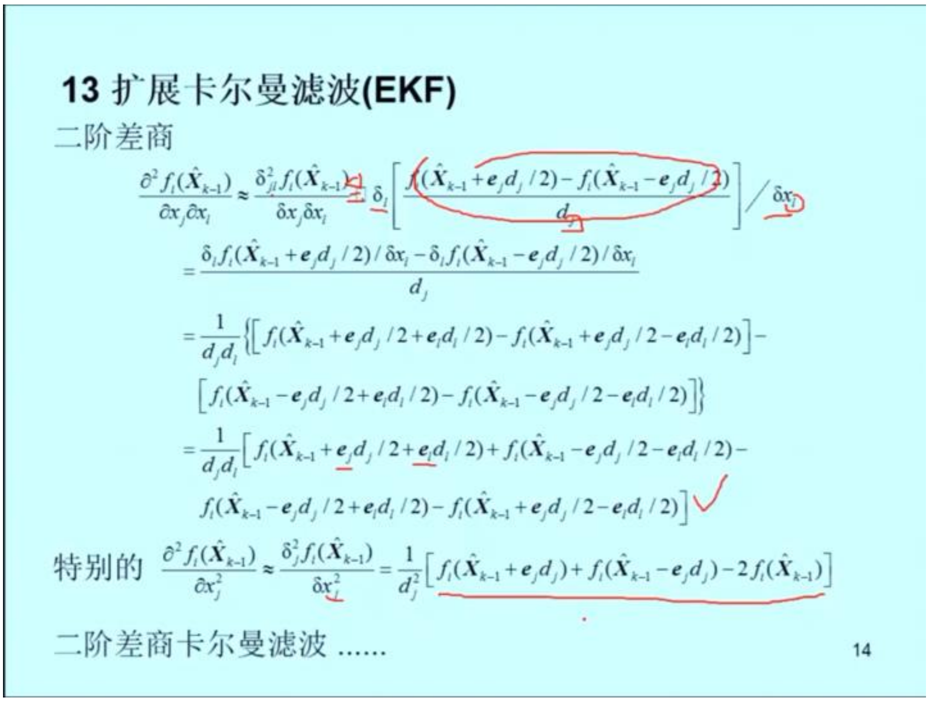
中心思想：当状态和量测不可导时，可以用差分代替微分



# 14 高斯分布采样非线性滤波

原函数积分 $\int_a^b f(x)\mathrm{d}x = F(b)-F(a)$

积分中值定理 $\int_a^b f(x)\mathrm{d}x = (b-a)f(\xi)$

## 14.1 高斯求积算法

经典矩形积分 $\int_a^b f(x)\mathrm{d}x \approx (b-a)f\left(\frac{a+b}{2}\right)$ □ $Rect$

经典梯形积分 $\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{2}\left[f(a)+f(b)\right]$ □ $Trap$

都只有1次代数精度！

数值积分形式 $\int_a^b f(x)\mathrm{d}x \approx \sum_{k=1}^{n} w_k f(x_k)$ $x_k$ 求积节点；$w_k$ 求积系数

**求积精度**：如果一个数值求积公式对于次数不大于m的多项式均能准确成立，但对于m+1次多项式不成立，则称之为具有m次代数精度。

$$\begin{cases}\sum_{k=1}^{n} w_k x_k^0 = \int_a^b 1\mathrm{d}x = b-a \\ \sum_{k=1}^{n} w_k x_k = \int_a^b x\mathrm{d}x = \frac{1}{2}\left(b^2-a^2\right) \\ \vdots \\ \sum_{k=1}^{n} w_k x_k^m = \int_a^b x^m\mathrm{d}x = \frac{1}{m+1}\left(b^{m+1}-a^{m+1}\right)\end{cases}$$

当积分节点**等间隔**选取即(b-a)/(n-1)时，n个节点的积分公式至少具有m=n-1次代数精度（当n为奇数时为n次代数精度）。

经典2点梯形公式、3点辛普森公式、5点科特斯公式都是等间隔的数值求积算法。



# 14 高斯分布采样非线性滤波

## 14.1 高斯求积算法

如果按**不等间隔**方式选取积分节点，则有可能使求积公式具有m=2n-1次代数精度，这类求积公式称为高斯求积公式，$x_k$称为高斯节点。

比如[-1,1]上的n次勒让德多项式有n个零点，它可作为n个高斯节点：

$P_0(x)=1$

$P_1(x)=x$

$P_2(x)=(3x^2-1)/2 \longrightarrow x_{1,2}=\mp 1/\sqrt{3}$

$P_3(x)=(5x^3-3x)/2$

$\cdots$

$$\begin{cases} w_1+w_2 = 1-(-1)=2 \\ w_1\left(-\frac{1}{\sqrt{3}}\right)+w_2\left(\frac{1}{\sqrt{3}}\right)=\frac{1}{2}\left[1^2-(-1)^2\right]=0\end{cases} \Longrightarrow w_1=w_2=1$$

得2点高斯求积算法 $\int_{-1}^{1} f(x)\mathrm{d}x \approx f\left(-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$ 具有3次代数精度！

验证：

$\int_{-1}^{1} x^0\mathrm{d}x = 2 = \left(-\frac{1}{\sqrt{3}}\right)^0+\left(\frac{1}{\sqrt{3}}\right)^0$ $\int_{-1}^{1} x^2\mathrm{d}x = \frac{2}{3} = \left(-\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2$ $\int_{-1}^{1} x^4\mathrm{d}x = \frac{2}{5} \neq \left(-\frac{1}{\sqrt{3}}\right)^4+\left(\frac{1}{\sqrt{3}}\right)^4 = \frac{2}{9}$

$\int_{-1}^{1} x^1\mathrm{d}x = 0 = \left(-\frac{1}{\sqrt{3}}\right)^1+\left(\frac{1}{\sqrt{3}}\right)^1$ $\int_{-1}^{1} x^3\mathrm{d}x = 0 = \left(-\frac{1}{\sqrt{3}}\right)^3+\left(\frac{1}{\sqrt{3}}\right)^3$

# 14 高斯分布采样非线性滤波

## 14.1 高斯求积算法

带权重的高斯求积公式 $\int_a^b \rho(x) f(x)\mathrm{d}x \approx \sum_{k=1}^{n} w_k f(x_k)$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\mathrm{d}x = \sum_{k=1}^{n} w_k f(x_k)$$ 高斯—切比雪夫(Chebyshev)求积

$$\int_{0}^{\infty} \mathrm{e}^{-x} f(x)\mathrm{d}x = \sum_{k=1}^{n} w_k f(x_k)$$ 高斯—拉盖尔(Laguerre)求积

$$\int_{0}^{\infty} x^{\alpha/2-1} \mathrm{e}^{-x} f(x)\mathrm{d}x = \sum_{k=1}^{n} w_k f(x_k)$$ 高斯—广义拉盖尔求积

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2} f(x)\mathrm{d}x = \sum_{k=1}^{n} w_k f(x_k)$$ 高斯—埃尔米特(Hermite)求积 $x_{1,2} = \pm 1$

| 勒让德多项式 | 切比雪夫多项式 | 广义拉盖尔多项式 | 埃尔米特多项式 |
|---|---|---|---|
| $P_0(x)=1$ | $T_0(x)=1$ | $L_0^\alpha(x)=1$ | $H_0(x)=1$ |
| $P_1(x)=x$ | $T_1(x)=x$ | $L_1^\alpha(x)=-x+\alpha+1$ | $H_1(x)=x$ |
| $P_2(x)=(3x^2-1)/2$ | $T_2(x)=2x^2-1$ | $L_2^\alpha(x)=x^2-(2\alpha+4)x+(\alpha^2+2)$ | $H_2(x)=x^2-1$ |
| $P_3(x)=(5x^3-3x)/2$ | $T_3(x)=4x^3-3x$ | $L_3^\alpha(x)=$ | $H_3(x)=x^3-3x$ |

# 14 高斯分布采样非线性滤波

## 14.2 高斯求积滤波

标量非线性概率函数 $Z=h(X)$ $p(x)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$ □ $\mathrm{N}(x\,|\,0,1)$ 标准高斯

$$\mathrm{E}[Z]=\int_{-\infty}^{\infty} h(x)p(x)\mathrm{d}x=\int_{-\infty}^{\infty} h(x)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)\mathrm{e}^{-x^2/2}\mathrm{d}x \approx \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{s} w_k h(x_k)$$

$$\mathrm{Var}[Z]=\int_{-\infty}^{\infty}\left(h(x)-\mathrm{E}[Z]\right)^2 p(x)\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(h(x)-\mathrm{E}[Z]\right)^2\mathrm{e}^{-x^2/2}\mathrm{d}x \approx \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{s} w_k\left(h(x_k)-\mathrm{E}[Z]\right)^2$$

若$X$高斯分布不标准 $X\sim\mathrm{N}(\mu,\sigma^2)$ ，作变换 $x'=(x-\mu)/\sigma$ ，则有

$$\mathrm{E}[Z]=\int_{-\infty}^{\infty} h(x)\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(\sigma x'+\mu)\mathrm{e}^{-x'^2/2}\mathrm{d}x'$$

$$\approx \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{s} w_k h(\sigma x'_k+\mu) \text{ □ } \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{s} w_k h(\sigma x_k+\mu)$$

$$\mathrm{Var}[Z]\approx\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{s} w_k\left(h(\sigma x_k+\mu)-\mathrm{E}[Z]\right)^2$$

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## 14 高斯分布采样非线性滤波

### 14.2 高斯求积滤波

$\{x_{k_1}\} = \left\{\begin{bmatrix}1\\ \vdots\end{bmatrix}, \begin{bmatrix}-1\\ \vdots\end{bmatrix}\right\}$　　$s=2, n=3$

$\{x_{k_1,k_2}\} = \left\{\begin{bmatrix}1\\1\\ \vdots\end{bmatrix}, \begin{bmatrix}-1\\1\\ \vdots\end{bmatrix}, \begin{bmatrix}1\\-1\\ \vdots\end{bmatrix}, \begin{bmatrix}-1\\-1\\ \vdots\end{bmatrix}\right\}$

$\{x_{k_1,k_2,k_3}\} = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\end{bmatrix}\right\}$

向量非线性函数 $Z = h(X)$　　$X \sim N(0, I)$

$$E[Z] = \int_{-\infty}^{\infty} h(x)p(x)dx = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)p(x_1)p(x_2)\cdots p(x_n)dx_1dx_2\cdots dx_n$$

└> x相互独立

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)p(x_1)dx_1 p(x_2)dx_2\cdots p(x_n)dx_n$$

$$\approx \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{k_1=1}^{s} w_{k_1} h(x_{k_1}) p(x_2)dx_2\cdots p(x_n)dx_n$$

$$\approx \frac{1}{(2\pi)^{2/2}}\int_{-\infty}^{\infty}\cdots\sum_{k_2=1}^{s} w_{k_2}\sum_{k_1=1}^{s} w_{k_1} h(x_{k_1,k_2})\cdots p(x_n)dx_n$$

$$\approx \frac{1}{(2\pi)^{n/2}}\sum_{k_n=1}^{s} w_{k_n}\cdots\sum_{k_2=1}^{s} w_{k_2}\sum_{k_1=1}^{s} w_{k_1} h(x_{k_1,k_2,\cdots,k_n}) \triangleq \frac{1}{(2\pi)^{n/2}}\sum_{k_1,k_2,\cdots,k_n=1}^{s^n} w_{k_1,k_2,\cdots,k_n} h(x_{k_1,k_2,\cdots,k_n})$$

高斯节点　$\{x_{k_1}\} = \left\{\begin{bmatrix}x_1\\ \vdots\end{bmatrix}, \begin{bmatrix}x_2\\ \vdots\end{bmatrix}, \cdots, \begin{bmatrix}x_s\\ \vdots\end{bmatrix}\right\}$

$\{x_{k_1,k_2}\} = \left\{\begin{bmatrix}x_1\\x_1\\ \vdots\end{bmatrix}, \begin{bmatrix}x_2\\x_1\\ \vdots\end{bmatrix}, \cdots, \begin{bmatrix}x_s\\x_1\\ \vdots\end{bmatrix}, \begin{bmatrix}x_1\\x_2\\ \vdots\end{bmatrix}, \begin{bmatrix}x_2\\x_2\\ \vdots\end{bmatrix}, \cdots, \begin{bmatrix}x_s\\x_2\\ \vdots\end{bmatrix}, \cdots, \begin{bmatrix}x_1\\x_s\\ \vdots\end{bmatrix}, \begin{bmatrix}x_2\\x_s\\ \vdots\end{bmatrix}, \cdots, \begin{bmatrix}x_s\\x_s\\ \vdots\end{bmatrix}\right\}$

⋮

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## 14 高斯分布采样非线性滤波

### 14.2 高斯求积滤波

$$\begin{cases} X_k = f(X_{k-1}) + \Gamma_{k-1}W_{k-1} \\ Z_k = h(X_k) + V_k \end{cases}$$

⇩

$$\begin{cases} K_k = P_{XZ,k/k-1}P_{ZZ,k/k-1}^{-1} \\ \hat{X}_k = \hat{X}_{k/k-1} + K_k(Z_k - \hat{Z}_{k/k-1}) \\ P_k = P_{k/k-1} - K_k P_{ZZ,k/k-1} K_k^T \end{cases}$$

滤波框架不变
和体求 $K_k, \hat{X}_k, P_k$ 变了

若$X$非标准高斯分布 $X \sim N(\mu, P)$，则有

$$E[Z] = \frac{1}{(2\pi)^{n/2}}\sum_{k_1,k_2,\cdots,k_n=1}^{s^n} w_{k_1,k_2,\cdots,k_n} h(\sqrt{P}x_{k_1,k_2,\cdots,k_n} + \mu)$$

GHQKF滤波的求积分计算过程：（特点：精度高但计算量大！）

└> 滤波一次精度高，一次迭代后不一定服从高斯分布了.

一阶矩 视为均值

$$\hat{X}_{k/k-1} = \frac{1}{(2\pi)^{n/2}}\sum_{k_1,k_2,\cdots,k_n=1}^{s^n} w_{k_1,k_2,\cdots,k_n} f(\sqrt{P_{k-1}}x_{k_1,k_2,\cdots,k_n} + \hat{X}_{k-1})$$

方差 ←

$$P_{k/k-1} = \frac{1}{(2\pi)^{n/2}}\sum_{k_1,k_2,\cdots,k_n=1}^{s^n} w_{k_1,k_2,\cdots,k_n} \left\langle f(\sqrt{P_{k-1}}x_{k_1,k_2,\cdots,k_n} + \hat{X}_{k-1}) - \hat{X}_{k/k-1}\right\rangle^2 + \Gamma Q_{k-1}\Gamma^T$$

方差 ←

$$\hat{Z}_{k/k-1} = \frac{1}{(2\pi)^{n/2}}\sum_{k_1,k_2,\cdots,k_n=1}^{s^n} w_{k_1,k_2,\cdots,k_n} h(\sqrt{P_{k/k-1}}x_{k_1,k_2,\cdots,k_n} + \hat{X}_{k/k-1})$$

$\langle M\rangle^2 \triangleq MM^T$

方差 ←

$$P_{ZZ,k/k-1} = \frac{1}{(2\pi)^{n/2}}\sum_{k_1,k_2,\cdots,k_n=1}^{s^n} w_{k_1,k_2,\cdots,k_n} \left\langle h(\sqrt{P_{k/k-1}}x_{k_1,k_2,\cdots,k_n} + \hat{X}_{k/k-1}) - \hat{Z}_{k/k-1}\right\rangle^2 + R_k$$

协方差 ←

$$P_{XZ,k/k-1} = \frac{1}{(2\pi)^{n/2}}\sum_{k_1,k_2,\cdots,k_n=1}^{s^n} w_{k_1,k_2,\cdots,k_n} \left(\sqrt{P_{k/k-1}}x_{k_1,k_2,\cdots,k_n} + \hat{X}_{k/k-1}\right)\left(h(\sqrt{P_{k/k-1}}x_{k_1,k_2,\cdots,k_n} + \hat{X}_{k/k-1}) - \hat{Z}_{k/k-1}\right)^T$$

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中心思想：泰勒展开后的微分量用差分代替

$$\boldsymbol{Y} = \tilde{\boldsymbol{f}}(\boldsymbol{Z}) = \tilde{\boldsymbol{f}}(\hat{\boldsymbol{Z}}) + \sum_{i=1}^{\infty} \frac{1}{i!} \boldsymbol{D}_{\Delta \boldsymbol{Z}}^{i} \tilde{\boldsymbol{f}}$$

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$$\boldsymbol{D}_{\Delta \boldsymbol{Z}}^{1} \tilde{\boldsymbol{f}} = \left( \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \Delta z_j \right) \tilde{\boldsymbol{f}}(\hat{\boldsymbol{Z}}) = \sum_{j=1}^{m} \left[ \boldsymbol{e}_j \left( \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \Delta z_i \right) \tilde{f}_j(\hat{\boldsymbol{Z}}) \right]$$

## 14.3 差分Kalman滤波

$$\boldsymbol{D}_{\Delta \boldsymbol{Z}} \tilde{\boldsymbol{f}} \approx \tilde{\boldsymbol{D}}_{\Delta \boldsymbol{Z}} \tilde{\boldsymbol{f}} \triangleq \sum_{j=1}^{m} \left[ \boldsymbol{e}_j \left( \sum_{i=1}^{n} \frac{\bar{\delta}_i}{2\delta z_i} \Delta z_i \right) \tilde{f}_j(\hat{\boldsymbol{Z}}) \right] = \sum_{j=1}^{m} \boldsymbol{e}_j \left( \sum_{i=1}^{n} \frac{\bar{\delta}_i \tilde{f}_j(\hat{\boldsymbol{Z}})}{2d_i} \Delta z_i \right)$$

差分代替微分

$$\boldsymbol{Y} = \tilde{\boldsymbol{f}}(\hat{\boldsymbol{Z}}) + \tilde{\boldsymbol{D}}_{\Delta \boldsymbol{Z}} \tilde{\boldsymbol{f}}$$

$$\bar{\delta}_i \tilde{f}_j(\hat{\boldsymbol{Z}}) = \tilde{f}_j(\hat{\boldsymbol{Z}} + \boldsymbol{e}_i d_i) - \tilde{f}_j(\hat{\boldsymbol{Z}} - \boldsymbol{e}_i d_i)$$

假设标准正态分布 $\Delta \boldsymbol{Z} \sim \mathrm{N}(\boldsymbol{0}, \mathbf{I})$，则输出的统计特性：

$$\bar{\boldsymbol{Y}} = \mathrm{E}\left[ \tilde{\boldsymbol{f}}(\hat{\boldsymbol{Z}}) + \tilde{\boldsymbol{D}}_{\Delta \boldsymbol{Z}} \tilde{\boldsymbol{f}} \right] = \tilde{\boldsymbol{f}}(\hat{\boldsymbol{Z}})$$

$$\boldsymbol{P}_Y = \mathrm{E}\left[ (\boldsymbol{Y} - \bar{\boldsymbol{Y}})(\boldsymbol{Y} - \bar{\boldsymbol{Y}})^{\mathrm{T}} \right] = \mathrm{E}\left[ (\tilde{\boldsymbol{D}}_{\Delta \boldsymbol{Z}} \tilde{\boldsymbol{f}})(\tilde{\boldsymbol{D}}_{\Delta \boldsymbol{Z}} \tilde{\boldsymbol{f}})^{\mathrm{T}} \right] = \mathrm{E}\left\{ \sum_{j=1}^{m} \boldsymbol{e}_j \left( \sum_{i=1}^{n} \frac{\delta_i \tilde{f}_j(\hat{\boldsymbol{Z}})}{2d_i} \Delta z_i \right) \left[ \sum_{l=1}^{m} \boldsymbol{e}_l \left( \sum_{k=1}^{n} \frac{\delta_k \tilde{f}_l(\hat{\boldsymbol{Z}})}{2d_k} \Delta z_k \right) \right]^{\mathrm{T}} \right\}$$

$$= \sum_{j=1}^{m} \sum_{l=1}^{m} \boldsymbol{e}_j \boldsymbol{e}_l^{\mathrm{T}} \mathrm{E}\left[ \left( \sum_{i=1}^{n} \frac{\delta_i \tilde{f}_j(\hat{\boldsymbol{Z}})}{2d_i} \Delta z_i \right) \left( \sum_{k=1}^{n} \frac{\delta_k \tilde{f}_l(\hat{\boldsymbol{Z}})}{2d_k} \Delta z_k \right) \right] \Box \sum_{j=1}^{m} \sum_{l=1}^{m} \boldsymbol{e}_j P_{Y,jl}$$

$$P_{Y,jl} = \mathrm{E}\left[ \left( \sum_{i=1}^{n} \frac{\delta_i \tilde{f}_j(\hat{\boldsymbol{Z}})}{2d_i} \Delta z_i \right) \left( \sum_{k=1}^{n} \frac{\delta_k \tilde{f}_l(\hat{\boldsymbol{Z}})}{2d_k} \Delta z_k \right) \right] = \sum_{i=1}^{n} \frac{\delta_i \tilde{f}_j(\hat{\boldsymbol{Z}}) \cdot \delta_i \tilde{f}_l(\hat{\boldsymbol{Z}})}{2d \cdot 2d} \mathrm{E}\left[ \Delta z_i \Delta z_i \right]$$

$$= \frac{1}{4d^2} \sum_{i=1}^{n} \left[ \tilde{f}_j(\hat{\boldsymbol{Z}} + \boldsymbol{e}_i d) - \tilde{f}_j(\hat{\boldsymbol{Z}} - \boldsymbol{e}_i d) \right] \left[ \tilde{f}_l(\hat{\boldsymbol{Z}} + \boldsymbol{e}_i d) - \tilde{f}_l(\hat{\boldsymbol{Z}} - \boldsymbol{e}_i d) \right]$$

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$$P_{Y,jl} = \frac{1}{4d^2} \sum_{i=1}^{n} \left[ \tilde{f}_j(\hat{\boldsymbol{Z}} + \boldsymbol{e}_i d) - \tilde{f}_j(\hat{\boldsymbol{Z}} - \boldsymbol{e}_i d) \right] \left[ \tilde{f}_l(\hat{\boldsymbol{Z}} + \boldsymbol{e}_i d) - \tilde{f}_l(\hat{\boldsymbol{Z}} - \boldsymbol{e}_i d) \right]$$


# 14 高斯分布采样非线性滤波

## 14.3 差分Kalman滤波


若分布不标准 $\Delta \boldsymbol{X} \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{P}_{\Delta X})$ 记 $\boldsymbol{P}_{\Delta X} = \boldsymbol{S}\boldsymbol{S}^{\mathrm{T}}$ $\quad \tilde{f}_j(\boldsymbol{Z}) = f_j(\boldsymbol{S}\boldsymbol{Z}) = f_j(\boldsymbol{X})$

则有 $\tilde{f}_j(\hat{\boldsymbol{Z}} \pm \boldsymbol{e}_i d) = f_j(\boldsymbol{S}(\hat{\boldsymbol{Z}} \pm \boldsymbol{e}_i d)) = f_j(\hat{\boldsymbol{X}} \pm \boldsymbol{S}_i d)$

$$P_{Y,jl} = \frac{1}{4d^2} \sum_{i=1}^{n} \left[ f_j(\hat{\boldsymbol{X}} + \boldsymbol{S}_i d) - f_j(\hat{\boldsymbol{X}} - \boldsymbol{S}_i d) \right] \left[ f_l(\hat{\boldsymbol{X}} + \boldsymbol{S}_i d) - f_l(\hat{\boldsymbol{X}} - \boldsymbol{S}_i d) \right]$$

$$\boldsymbol{P}_Y = \frac{1}{4d^2} \sum_{i=1}^{n} \left[ \boldsymbol{f}(\hat{\boldsymbol{X}} + \boldsymbol{S}_i d) - \boldsymbol{f}(\hat{\boldsymbol{X}} - \boldsymbol{S}_i d) \right] \left[ \boldsymbol{f}(\hat{\boldsymbol{X}} + \boldsymbol{S}_i d) - \boldsymbol{f}(\hat{\boldsymbol{X}} - \boldsymbol{S}_i d) \right]^{\mathrm{T}}$$

$$\boldsymbol{P}_{XY} = \mathrm{E}\left[ (\boldsymbol{X} - \bar{\boldsymbol{X}})(\boldsymbol{Y} - \bar{\boldsymbol{Y}})^{\mathrm{T}} \right] = \mathrm{E}\left[ (\Delta \boldsymbol{X})(\tilde{\boldsymbol{D}}_{\Delta \boldsymbol{Z}} \tilde{\boldsymbol{f}})^{\mathrm{T}} \right]$$

$$= \frac{1}{4d^2} \sum_{i=1}^{n} \left[ (\hat{\boldsymbol{X}} + \boldsymbol{S}_i d) - (\hat{\boldsymbol{X}} - \boldsymbol{S}_i d) \right] \left[ \boldsymbol{f}(\hat{\boldsymbol{X}} + \boldsymbol{S}_i d) - \boldsymbol{f}(\hat{\boldsymbol{X}} - \boldsymbol{S}_i d) \right]^{\mathrm{T}}$$

$$= \frac{1}{2d} \sum_{i=1}^{n} \boldsymbol{S}_i \left[ \boldsymbol{f}(\hat{\boldsymbol{X}} + \boldsymbol{S}_i d) - \boldsymbol{f}(\hat{\boldsymbol{X}} - \boldsymbol{S}_i d) \right]^{\mathrm{T}}$$

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# 14 高斯分布采样非线性滤波

## 14.3 差分Kalman滤波

$$\begin{cases} \boldsymbol{X}_k = \boldsymbol{f}(\boldsymbol{X}_{k-1}) + \boldsymbol{\Gamma}_{k-1}\boldsymbol{W}_{k-1} \\ \boldsymbol{Z}_k = \boldsymbol{h}(\boldsymbol{X}_k) + \boldsymbol{V}_k \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \boldsymbol{K}_k = \boldsymbol{P}_{XZ,k/k-1}\boldsymbol{P}_{ZZ,k/k-1}^{-1} \\ \hat{\boldsymbol{X}}_k = \hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{K}_k(\boldsymbol{Z}_k - \hat{\boldsymbol{Z}}_{k/k-1}) \\ \boldsymbol{P}_k = \boldsymbol{P}_{k/k-1} - \boldsymbol{K}_k\boldsymbol{P}_{ZZ,k/k-1}\boldsymbol{K}_k^{\mathrm{T}} \end{cases}$$

（1）一阶差分CDKF

$$\hat{\boldsymbol{X}}_{k/k-1} = \boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1})$$

$$\boldsymbol{P}_{k/k-1} = \frac{1}{4d^2}\sum_{i=1}^{n}\left\langle \boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1} + \boldsymbol{S}_{k-1,i}d) - \boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1} - \boldsymbol{S}_{k-1,i}d)\right\rangle^2 + \boldsymbol{\Gamma}_{k-1}\boldsymbol{Q}_{k-1}\boldsymbol{\Gamma}_{k-1}^{\mathrm{T}}$$

$$\boldsymbol{S}_{k-1} = \sqrt{\boldsymbol{P}_{k-1}}$$

$$\hat{\boldsymbol{Z}}_{k/k-1} = \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1})$$

$$\boldsymbol{P}_{ZZ,k/k-1} = \frac{1}{4d^2}\sum_{i=1}^{n}\left\langle \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{S}_{k/k-1,i}d) - \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} - \boldsymbol{S}_{k/k-1,i}d)\right\rangle^2 + \boldsymbol{R}_k$$

$$\boldsymbol{S}_{k/k-1} = \sqrt{\boldsymbol{P}_{k/k-1}}$$

$$\boldsymbol{P}_{XZ,k/k-1} = \frac{1}{2d}\sum_{i=1}^{n}\boldsymbol{S}_{k/k-1,i}\left[\boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{S}_{k/k-1,i}d) - \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} - \boldsymbol{S}_{k/k-1,i}d)\right]^{\mathrm{T}}$$

相较于DQEKF，CDKF非线性函数计算次数少，但需chol分解。

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# 14 高斯分布采样非线性滤波

## 14.3 差分Kalman滤波

$$\begin{cases} \boldsymbol{X}_k = \boldsymbol{f}(\boldsymbol{X}_{k-1}) + \boldsymbol{\Gamma}_{k-1}\boldsymbol{W}_{k-1} \\ \boldsymbol{Z}_k = \boldsymbol{h}(\boldsymbol{X}_k) + \boldsymbol{V}_k \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \boldsymbol{K}_k = \boldsymbol{P}_{XZ,k/k-1}\boldsymbol{P}_{ZZ,k/k-1}^{-1} \\ \hat{\boldsymbol{X}}_k = \hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{K}_k(\boldsymbol{Z}_k - \hat{\boldsymbol{Z}}_{k/k-1}) \\ \boldsymbol{P}_k = \boldsymbol{P}_{k/k-1} - \boldsymbol{K}_k\boldsymbol{P}_{ZZ,k/k-1}\boldsymbol{K}_k^{\mathrm{T}} \end{cases}$$

（2）二阶差分CDKF

$$\hat{\boldsymbol{X}}_{k/k-1} = \frac{d^2-n}{d^2}\boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1}) + \frac{1}{2d^2}\sum_{i=1}^{n}\left[\boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1} + \boldsymbol{S}_{k-1,i}d) + \boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1} - \boldsymbol{S}_{k-1,i}d)\right]$$

$$\boldsymbol{P}_{k/k-1} = \frac{1}{4d^2}\sum_{i=1}^{n}\left\langle \boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1} + \boldsymbol{S}_{k-1,i}d) - \boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1} - \boldsymbol{S}_{k-1,i}d)\right\rangle^2 + \frac{d^2-1}{4d^2}\sum_{i=1}^{n}\left\langle \boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1} + \boldsymbol{S}_{k-1,i}d) + \boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1} - \boldsymbol{S}_{k-1,i}d) - 2\boldsymbol{f}(\hat{\boldsymbol{X}}_{k-1})\right\rangle^2 + \boldsymbol{\Gamma}_{k-1}\boldsymbol{Q}_{k-1}\boldsymbol{\Gamma}_{k-1}^{\mathrm{T}}$$

与二阶EKF相比，计算量大幅减小。

$$\hat{\boldsymbol{Z}}_{k/k-1} = \frac{d^2-n}{d^2}\boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1}) + \frac{1}{2d^2}\sum_{i=1}^{n}\left[\boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{S}_{k/k-1,i}d) + \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} - \boldsymbol{S}_{k/k-1,i}d)\right]$$

$$\boldsymbol{P}_{ZZ,k/k-1} = \frac{1}{4d^2}\sum_{i=1}^{n}\left\langle \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{S}_{k/k-1,i}d) - \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} - \boldsymbol{S}_{k/k-1,i}d)\right\rangle^2 + \frac{d^2-1}{4d^2}\sum_{i=1}^{n}\left\langle \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{S}_{k/k-1,i}d) + \boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1} - \boldsymbol{S}_{k/k-1,i}d) - 2\boldsymbol{h}(\hat{\boldsymbol{X}}_{k/k-1})\right\rangle^2 + \boldsymbol{R}_k$$

$$\boldsymbol{P}_{XZ,k/k-1} = \frac{1}{2d}\sum_{i=1}^{n}\boldsymbol{S}_{k/k-1,i}\left[\boldsymbol{h}(\hat{\boldsymbol{X}}_{k-1} + \boldsymbol{S}_{k/k-1,i}d) - \boldsymbol{h}(\hat{\boldsymbol{X}}_{k-1} - \boldsymbol{S}_{k/k-1,i}d)\right]^{\mathrm{T}}$$

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# 14 高斯分布采样非线性滤波

## 14.4 容积卡尔曼滤波（CKF）

（1）球面—径向积分准则

$$\iiint_{\mathbb{R}^3} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^\infty \int_0^{2\pi} \int_0^\pi f(r\mathbf{s}) r^2 \sin\varphi \mathrm{d}\varphi \mathrm{d}\theta \mathrm{d}r = \int_0^\infty r^2 \int_{U_2} f(r\mathbf{s})\mathrm{d}\sigma(\mathbf{s})\mathrm{d}r$$

所谓多积就是高维空间求导.

针对高斯加权积分 $I(f) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\exp(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})\mathrm{d}\boldsymbol{x}$，令 $\boldsymbol{x} = r\boldsymbol{s}$ $\quad \boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} = 1, r = \sqrt{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$

可转换为球面—径向坐标积分：

$$I(f) = \int_0^\infty \int_{U_n} f(r\boldsymbol{s}) r^{n-1} \mathrm{e}^{-r^2} \mathrm{d}\sigma(\boldsymbol{s})\mathrm{d}r = \int_0^\infty r^{n-1}\mathrm{e}^{-r^2} \int_{U_n} f(r\boldsymbol{s})\mathrm{d}\sigma(\boldsymbol{s})\mathrm{d}r$$

$$S(r) = \int_{U_n} f(r\boldsymbol{s})\mathrm{d}\sigma(\boldsymbol{s}) = \sum_{i=1}^{n_s} w_{s,i} f(r\boldsymbol{s}_i)$$

$$I(f) = \int_0^\infty S(r) r^{n-1}\mathrm{e}^{-r^2}\mathrm{d}r = \sum_{j=1}^{n_r} w_{r,j} S(r_j)$$

$$= \sum_{j=1}^{n_r} w_{r,j} \sum_{i=1}^{n_s} w_{s,i} f(r_j \boldsymbol{s}_i) = \sum_{j=1}^{n_r}\sum_{i=1}^{n_s} w_{r,j} w_{s,i} f(r_j \boldsymbol{s}_i)$$

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# 14 高斯分布采样非线性滤波

## 14.4 容积卡尔曼滤波（CKF）

$$S(r) = \int_{U_n} f(r\boldsymbol{s})\mathrm{d}\sigma(\boldsymbol{s})$$

$$\begin{cases} \Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{t}dt \ (x>0) \\ \Gamma(x+1) = x\Gamma(x), \Gamma(1/2) = \sqrt{\pi} \end{cases}$$

A)球面积分准则

单位半径球面上的单项式积分公式：

$$S_d = \int_{U_n} s_1^{d_1} s_2^{d_2} \cdots s_n^{d_n} \mathrm{d}\sigma(\boldsymbol{s}) = 2\frac{\prod_{i=1}^n \Gamma((d_i+1)/2)}{\Gamma((d+n)/2)} \quad ✓ \qquad d = \sum_{i=1}^n d_i$$

当d=odd（奇数）时，单项式 $s_1^{d_1} s_2^{d_2} \cdots s_n^{d_n}$ 在单位球面上正负对称，总有 $S_{odd} = 0$

当总阶次d=0时有任意$d_i$=0，有

$$S_0 = \int_{U_n} 1\mathrm{d}\sigma(\boldsymbol{s}) = 2\frac{\prod_{i=1}^n \Gamma(1/2)}{\Gamma((0+n)/2)} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \triangleq A_n$$

特例

$A_1 = 2\pi^{1/2} / \Gamma(1/2) = 2\pi^{1/2} / \pi^{1/2} = 2$ → 直径（一维空间）

$A_2 = 2\pi^{2/2} / \Gamma(2/2) = 2\pi / 1 = 2\pi$ → 周长（二维）

$A_3 = 2\pi^{3/2} / \Gamma(3/2) = 2\pi^{3/2} / (\pi^{1/2}/2) = 4\pi$ → 表面积（三维）

当总阶次d=2时

若$d_i$=$d_j$=1，有 $\quad S_{2(i\neq j)} = \int_{U_n} s_i s_j \mathrm{d}\sigma(\boldsymbol{s}) = 0$

若$d_i$=2，有

$$S_2 = S_{2(i=j)} = \int_{U_n} s_i^2 \mathrm{d}\sigma(\boldsymbol{s}) = 2\frac{\Gamma(3/2)\cdot\prod_{j=1,j\neq i}^n \Gamma(1/2)}{\Gamma((2+n)/2)} = 2\frac{\sqrt{\pi}/2 \times \sqrt{\pi}^{n-1}}{\Gamma(n/2)\times n/2} = \frac{A_n}{n}$$

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## 14 高斯分布采样非线性滤波

### 14.4 容积卡尔曼滤波（CKF）

$$S=\int_{U_n} f(\mathbf{s})\mathrm{d}\sigma(\mathbf{s})\approx w_s\sum_{i=1}^{2n} f(\mathbf{s}_i)$$

$$\approx \frac{A_n}{2n}\sum_{i=1}^{n}[f(\mathbf{e}_i)+f(-\mathbf{e}_i)]$$

令 $f(\mathbf{s})=1$，$f(\mathbf{s})=s_i^2=f(\pm\mathbf{s}_i)$，得

$$S_0=A_n\approx w_s\sum_{i=1}^{2n}1=2nw_s$$

$$S_2=A_n/n\approx w_s[f(\mathbf{s}_i)+f(-\mathbf{s}_i)]=2w_s s_i^2$$

$$\Rightarrow\quad w_s=\frac{A_n}{2n}=\frac{\pi^{n/2}}{n\Gamma(n/2)}\qquad s_i^2=1$$

球面积分方法与球面半径无关 $f(\mathbf{s})\to f(r\mathbf{s})$

$$S_0(r)=\int_{U_n} r\mathrm{d}\sigma(\mathbf{s})=rA_n\approx w_s\sum_{i=1}^{2n}r=2nrw_s$$

$$S_2(r)=\int_{U_n} r^2 s_i^2\mathrm{d}\sigma(\mathbf{s})=r^2A_n/n\approx w_s[f(r\mathbf{s}_i)+f(-r\mathbf{s}_i)]=2w_s r^2 s_i^2$$

$$\Rightarrow\quad S(r)=\int_{U_n} f(r\mathbf{s})\mathrm{d}\sigma(\mathbf{s})\approx\frac{A_n}{2n}\sum_{i=1}^{n}[f(r\mathbf{e}_i)+f(-r\mathbf{e}_i)]$$

→高斯积分转到球面上后，高斯点为2n个，运算减少

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## 14 高斯分布采样非线性滤波

### 14.4 容积卡尔曼滤波（CKF）

B)径向积分准则

$$I(f)=\int_0^\infty S(r)r^{n-1}\mathrm{e}^{-r^2}\mathrm{d}r=\int_0^\infty S(\sqrt{t})t^{n/2-1}\mathrm{e}^{-t}\mathrm{d}t\qquad t=r^2$$

$$\approx\Gamma(n/2)S(\sqrt{n/2})\;\square\; w_{r,1}S(r_1)$$

1点广义高斯—拉盖尔求积，具有1次代数精度（对于r为2次）

C)3阶球面—径向积分准则（具有3次代数精度）

$$I(f)=\int_{\square^n} f(\mathbf{x})\exp(-\mathbf{x}^{\mathrm{T}}\mathbf{x})\mathrm{d}\mathbf{x}\approx\sum_{j=1}^{n_r}\sum_{i=1}^{n_s}w_{r,j}w_{s,i}f(r_j\mathbf{s}_i)=w_{r,1}\sum_{i=1}^{2n}w_s f(r_1\mathbf{s}_i)$$

$$=\Gamma(n/2)\frac{\pi^{n/2}}{n\Gamma(n/2)}\sum_{i=1}^{2n}f\left(\sqrt{\frac{n}{2}}\mathbf{s}_i\right)=\frac{\pi^{n/2}}{n}\sum_{i=1}^{n}\left[f\left(\sqrt{\frac{n}{2}}\mathbf{e}_i\right)+f\left(-\sqrt{\frac{n}{2}}\mathbf{e}_i\right)\right]$$

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# 14 高斯分布采样非线性滤波

## 14.4 容积卡尔曼滤波（CKF）

对于非标准高斯分布$\mathrm{N}(\boldsymbol{x};\hat{\boldsymbol{X}},\boldsymbol{P})$，作变换 $\boldsymbol{x}=(2\boldsymbol{P})^{1/2}\boldsymbol{y}+\hat{\boldsymbol{X}}$ ，则有

$$\int_{\mathbb{R}^n}\boldsymbol{f}(\boldsymbol{x})\mathrm{N}(\boldsymbol{x};\hat{\boldsymbol{X}},\boldsymbol{P})\mathrm{d}\boldsymbol{x}=\int_{\mathbb{R}^n}\boldsymbol{f}\left((2\boldsymbol{P})^{1/2}\boldsymbol{y}+\hat{\boldsymbol{X}}\right)\frac{1}{(2\pi)^{n/2}|\boldsymbol{P}|^{1/2}}\exp(-\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y})|2\boldsymbol{P}|^{1/2}\mathrm{d}\boldsymbol{y}$$

$$=\frac{\sqrt{2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\boldsymbol{f}\left((2\boldsymbol{P})^{1/2}\boldsymbol{y}+\hat{\boldsymbol{X}}\right)\exp(-\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

$$=\frac{\sqrt{2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\boldsymbol{f}\left((2\boldsymbol{P})^{1/2}\boldsymbol{x}+\hat{\boldsymbol{X}}\right)\exp(-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

$$\approx\frac{\sqrt{2}}{(2\pi)^{n/2}}\frac{\pi^{n/2}}{n}\sum_{i=1}^{n}\left[\boldsymbol{f}\left(\sqrt{\frac{n}{2}}(2\boldsymbol{P})^{1/2}\boldsymbol{e}_i+\hat{\boldsymbol{X}}\right)+\boldsymbol{f}\left(-\sqrt{\frac{n}{2}}(2\boldsymbol{P})^{1/2}\boldsymbol{e}_i+\hat{\boldsymbol{X}}\right)\right]$$

$$=\frac{1}{2n}\sum_{i=1}^{n}\left[\boldsymbol{f}\left(\sqrt{n\boldsymbol{P}}\boldsymbol{e}_i+\hat{\boldsymbol{X}}\right)+\boldsymbol{f}\left(-\sqrt{n\boldsymbol{P}}\boldsymbol{e}_i+\hat{\boldsymbol{X}}\right)\right]$$ ✓

$$\begin{cases}w_i=1/(2n)\\\{\xi_i\}=\left\{\sqrt{n\boldsymbol{P}}+\left[\hat{\boldsymbol{X}}\right]_n\quad-\sqrt{n\boldsymbol{P}}+\left[\hat{\boldsymbol{X}}\right]_n\right\}\end{cases}$$



# 14 高斯分布采样非线性滤波

## 14.4 容积卡尔曼滤波（CKF）

$$\begin{cases}\boldsymbol{X}_k=\boldsymbol{f}(\boldsymbol{X}_{k-1})+\boldsymbol{\Gamma}_{k-1}\boldsymbol{W}_{k-1}\\\boldsymbol{Z}_k=\boldsymbol{h}(\boldsymbol{X}_k)+\boldsymbol{V}_k\end{cases}$$

⇩

$$\begin{cases}\boldsymbol{K}_k=\boldsymbol{P}_{XZ,k/k-1}\boldsymbol{P}_{ZZ,k/k-1}^{-1}\\\hat{\boldsymbol{X}}_k=\hat{\boldsymbol{X}}_{k/k-1}+\boldsymbol{K}_k(\boldsymbol{Z}_k-\hat{\boldsymbol{Z}}_{k/k-1})\\\boldsymbol{P}_k=\boldsymbol{P}_{k/k-1}-\boldsymbol{K}_k\boldsymbol{P}_{ZZ,k/k-1}\boldsymbol{K}_k^{\mathrm{T}}\end{cases}$$

（2）CKF滤波 →3阶精度

$$\hat{\boldsymbol{X}}_{k/k-1}=\sum_{i=1}^{L}w_i\boldsymbol{f}\left(\boldsymbol{\xi}_{i,k-1}\right)$$

$$\boldsymbol{P}_{k/k-1}=\sum_{i=1}^{L}w_i\left[\boldsymbol{f}\left(\boldsymbol{\xi}_{i,k-1}\right)-\hat{\boldsymbol{X}}_{k/k-1}\right]\left[\boldsymbol{f}\left(\boldsymbol{\xi}_{i,k-1}\right)-\hat{\boldsymbol{X}}_{k/k-1}\right]^{\mathrm{T}}+\boldsymbol{\Gamma}_{k-1}\boldsymbol{Q}_{k-1}\boldsymbol{\Gamma}_{k-1}^{\mathrm{T}}$$

$$\hat{\boldsymbol{Z}}_{k/k-1}=\sum_{i=1}^{L}w_i\boldsymbol{h}\left(\boldsymbol{\xi}_{i,k/k-1}\right)$$

$$\boldsymbol{P}_{ZZ,k/k-1}=\sum_{i=1}^{L}w_i\left[\boldsymbol{h}\left(\boldsymbol{\xi}_{i,k/k-1}\right)-\hat{\boldsymbol{Z}}_{k/k-1}\right]\left[\boldsymbol{h}\left(\boldsymbol{\xi}_{i,k/k-1}\right)-\hat{\boldsymbol{Z}}_{k/k-1}\right]^{\mathrm{T}}+\boldsymbol{R}_k$$

$$\boldsymbol{P}_{XZ,k/k-1}=\sum_{i=1}^{L}w_i\left(\boldsymbol{\xi}_{i,k/k-1}-\hat{\boldsymbol{X}}_{k/k-1}\right)\left[\boldsymbol{h}\left(\boldsymbol{\xi}_{i,k/k-1}\right)-\hat{\boldsymbol{Z}}_{k/k-1}\right]^{\mathrm{T}}$$

$$L=2n\qquad w_i=1/(2n)\qquad\{\xi_{i,K}\}=\left\{\sqrt{n\boldsymbol{P}_K}+\left[\hat{\boldsymbol{X}}_K\right]_n\quad-\sqrt{n\boldsymbol{P}_K}+\left[\hat{\boldsymbol{X}}_K\right]_n\right\}_{2n}\qquad K\,\square\,k-1,k/k-1$$

## 14 高斯分布采样非线性滤波

$$\begin{cases} \boldsymbol{X}_k = \boldsymbol{f}(\boldsymbol{X}_{k-1}) + \boldsymbol{\Gamma}_{k-1}\boldsymbol{W}_{k-1} \\ \boldsymbol{Z}_k = \boldsymbol{h}(\boldsymbol{X}_k) + \boldsymbol{V}_k \end{cases}$$

⇩

$$\begin{cases} \boldsymbol{K}_k = \boldsymbol{P}_{XZ,k/k-1}\boldsymbol{P}_{ZZ,k/k-1}^{-1} \\ \hat{\boldsymbol{X}}_k = \hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{K}_k(\boldsymbol{Z}_k - \hat{\boldsymbol{Z}}_{k/k-1}) \\ \boldsymbol{P}_k = \boldsymbol{P}_{k/k-1} - \boldsymbol{K}_k\boldsymbol{P}_{ZZ,k/k-1}\boldsymbol{K}_k^{\mathrm{T}} \end{cases}$$

### 14.4 容积卡尔曼滤波（CKF）

（3）5阶球面—径向容积准则CKF滤波

$$\hat{\boldsymbol{X}}_{k/k-1} = \sum_{i=1}^{L} w_i \boldsymbol{f}(\boldsymbol{\xi}_{i,k-1})$$

$$\boldsymbol{P}_{k/k-1} = \sum_{i=1}^{L} w_i \left[\boldsymbol{f}(\boldsymbol{\xi}_{i,k-1}) - \hat{\boldsymbol{X}}_{k/k-1}\right]\left[\boldsymbol{f}(\boldsymbol{\xi}_{i,k-1}) - \hat{\boldsymbol{X}}_{k/k-1}\right]^{\mathrm{T}} + \boldsymbol{\Gamma}_{k-1}\boldsymbol{Q}_{k-1}\boldsymbol{\Gamma}_{k-1}^{\mathrm{T}}$$

$$\hat{\boldsymbol{Z}}_{k/k-1} = \sum_{i=1}^{L} w_i \boldsymbol{h}(\boldsymbol{\xi}_{i,k/k-1})$$

$$\boldsymbol{P}_{ZZ,k/k-1} = \sum_{i=1}^{L} w_i \left[\boldsymbol{h}(\boldsymbol{\xi}_{i,k/k-1}) - \hat{\boldsymbol{Z}}_{k/k-1}\right]\left[\boldsymbol{h}(\boldsymbol{\xi}_{i,k/k-1}) - \hat{\boldsymbol{Z}}_{k/k-1}\right]^{\mathrm{T}} + \boldsymbol{R}_k$$

$$\boldsymbol{P}_{XZ,k/k-1} = \sum_{i=1}^{L} w_i \left(\boldsymbol{\xi}_{i,k/k-1} - \hat{\boldsymbol{X}}_{k/k-1}\right)\left[\boldsymbol{h}(\boldsymbol{\xi}_{i,k/k-1}) - \hat{\boldsymbol{Z}}_{k/k-1}\right]^{\mathrm{T}}$$

$$L = 2n^2+1 \quad \{w_i\} = \left\{\left[\frac{4-n}{2(n+2)^2}\right]_{2n} \left[\frac{1}{(n+2)^2}\right]_{2n(n-1)} \frac{2}{n+2}\right\} \quad \boldsymbol{e}_i^+ = (\boldsymbol{e}_j + \boldsymbol{e}_k)/\sqrt{2}, \boldsymbol{e}_i^- = (\boldsymbol{e}_j - \boldsymbol{e}_k)/\sqrt{2}$$

$$\{\boldsymbol{\xi}_{i,K}\} = \left\{\left[\hat{\boldsymbol{X}}_K\right]_{2n^2+1} + \sqrt{(n+2)\boldsymbol{P}_K}\left[\{\boldsymbol{e}_i\} \ \{-\boldsymbol{e}_i\} \ \{\boldsymbol{e}_i^+\} \ \{-\boldsymbol{e}_i^+\} \ \{\boldsymbol{e}_i^-\} \ \{-\boldsymbol{e}_i^-\} \ \{\boldsymbol{0}\}\right]\right\} \quad K \square k-1, k/k-1$$

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## 14 高斯分布采样非线性滤波

$$\begin{cases} \boldsymbol{X}_k = \boldsymbol{f}(\boldsymbol{X}_{k-1}) + \boldsymbol{\Gamma}_{k-1}\boldsymbol{W}_{k-1} \\ \boldsymbol{Z}_k = \boldsymbol{h}(\boldsymbol{X}_k) + \boldsymbol{V}_k \end{cases}$$

⇩

$$\begin{cases} \boldsymbol{K}_k = \boldsymbol{P}_{XZ,k/k-1}\boldsymbol{P}_{ZZ,k/k-1}^{-1} \\ \hat{\boldsymbol{X}}_k = \hat{\boldsymbol{X}}_{k/k-1} + \boldsymbol{K}_k(\boldsymbol{Z}_k - \hat{\boldsymbol{Z}}_{k/k-1}) \\ \boldsymbol{P}_k = \boldsymbol{P}_{k/k-1} - \boldsymbol{K}_k\boldsymbol{P}_{ZZ,k/k-1}\boldsymbol{K}_k^{\mathrm{T}} \end{cases}$$

### 14.5 UKF滤波

$$\hat{\boldsymbol{X}}_{k/k-1} = \sum_{i=1}^{L} w_i^m \boldsymbol{f}(\boldsymbol{\xi}_{i,k-1})$$

$$\boldsymbol{P}_{k/k-1} = \sum_{i=1}^{L} w_i^c \left[\boldsymbol{f}(\boldsymbol{\xi}_{i,k-1}) - \hat{\boldsymbol{X}}_{k/k-1}\right]\left[\boldsymbol{f}(\boldsymbol{\xi}_{i,k-1}) - \hat{\boldsymbol{X}}_{k/k-1}\right]^{\mathrm{T}} + \boldsymbol{\Gamma}_{k-1}\boldsymbol{Q}_{k-1}\boldsymbol{\Gamma}_{k-1}^{\mathrm{T}}$$

$$\hat{\boldsymbol{Z}}_{k/k-1} = \sum_{i=1}^{L} w_i^m \boldsymbol{h}(\boldsymbol{\xi}_{i,k/k-1})$$

$$\boldsymbol{P}_{ZZ,k/k-1} = \sum_{i=1}^{L} w_i^c \left[\boldsymbol{h}(\boldsymbol{\xi}_{i,k/k-1}) - \hat{\boldsymbol{Z}}_{k/k-1}\right]\left[\boldsymbol{h}(\boldsymbol{\xi}_{i,k/k-1}) - \hat{\boldsymbol{Z}}_{k/k-1}\right]^{\mathrm{T}} + \boldsymbol{R}_k$$

$$\boldsymbol{P}_{XZ,k/k-1} = \sum_{i=1}^{L} w_i^c \left(\boldsymbol{\xi}_{i,k/k-1} - \hat{\boldsymbol{X}}_{k/k-1}\right)\left[\boldsymbol{h}(\boldsymbol{\xi}_{i,k/k-1}) - \hat{\boldsymbol{Z}}_{k/k-1}\right]^{\mathrm{T}}$$

$$\{\xi_{i,K}\} = \left\{\gamma\sqrt{\boldsymbol{P}_K} + \left[\hat{\boldsymbol{X}}_K\right]_n \quad -\gamma\sqrt{\boldsymbol{P}_K} + \left[\hat{\boldsymbol{X}}_K\right]_n \quad \hat{\boldsymbol{X}}_K\right\} \qquad K \square k-1, k/k-1$$

$$\{w_i^m\} = \left\{\left[1/(2\gamma^2)\right]_{2n} \quad \lambda/\gamma^2\right\} \qquad \{w_i^c\} = \left\{\left[1/(2\gamma^2)\right]_{2n} \quad \lambda/\gamma^2 + (1-\alpha^2+\beta)\right\}$$

超参数 $\gamma = \sqrt{n+\lambda}, \lambda = \alpha^2(n+\kappa) - n \quad 10^{-4} \le \alpha \le 1, \kappa = 0, \beta \ge 0 (= 2)$

特例 $\alpha = 1, \kappa = \beta = 0 \Rightarrow \{w_i^c\} = \{[1/(2n)]_{2n}, 0\} \Leftrightarrow \mathrm{CKF}$

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## 14 高斯分布采样非线性滤波

### 14.6 SR-UKF滤波（平方根UKF）

$$\begin{cases} X_k = f(X_{k-1}) + \Gamma_{k-1}W_{k-1} \\ Z_k = h(X_k) + V_k \end{cases}$$

$$\begin{cases} K_k = P_{XZ,k/k-1}P_{ZZ,k/k-1}^{-1} \\ \hat{X}_k = \hat{X}_{k/k-1} + K_k(Z_k - \hat{Z}_{k/k-1}) \\ P_k = P_{k/k-1} - K_k P_{ZZ,k/k-1} K_k^{\mathrm{T}} \end{cases}$$

$$P_{k/k-1} = \sum_{i=1}^{L} w_i^c \left[f(\xi_{i,k-1}) - \hat{X}_{k/k-1}\right]\left[f(\xi_{i,k-1}) - \hat{X}_{k/k-1}\right]^{\mathrm{T}} + \Gamma_{k-1}Q_{k-1}\Gamma_{k-1}^{\mathrm{T}}$$

$$P_{ZZ,k/k-1} = \sum_{i=1}^{L} w_i^c \left[h(\xi_{i,k/k-1}) - \hat{Z}_{k/k-1}\right]\left[h(\xi_{i,k/k-1}) - \hat{Z}_{k/k-1}\right]^{\mathrm{T}} + R_k$$

$$\{w_i^c\} = \{[1/(2\gamma^2)]_{2n} \quad \lambda/\gamma^2 + (1-\alpha^2+\beta)\} \qquad \gamma = \sqrt{n+\lambda}, \lambda = \alpha^2(n+\kappa) - n \qquad 10^{-4} \le \alpha \le 1, \kappa = 0, \beta \ge 0(=2)$$

$$P_{k/k-1} = \left[\sqrt{w_1^c}\Delta\hat{X}_{1:2n}\right]\left[\sqrt{w_1^c}\Delta\hat{X}_{1:2n}\right]^{\mathrm{T}} + \Gamma_{k-1}\sqrt{Q_{k-1}}(\Gamma_{k-1}\sqrt{Q_{k-1}})^{\mathrm{T}} + w_{2n+1}^c\Delta\hat{X}_{2n+1}\Delta\hat{X}_{2n+1}^{\mathrm{T}}$$

$$= \left[\sqrt{w_1^c}\Delta\hat{X}_{1:2n} \quad \Gamma_{k-1}\sqrt{Q_{k-1}}\right]\left[\sqrt{w_1^c}\Delta\hat{X}_{1:2n} \quad \Gamma_{k-1}\sqrt{Q_{k-1}}\right]^{\mathrm{T}} - \sqrt{|w_{2n+1}^c|}\Delta\hat{X}_{2n+1}(\sqrt{|w_{2n+1}^c|}\Delta\hat{X}_{2n+1})^{\mathrm{T}}$$

QR分解

$$= R^{\mathrm{T}}R - \sqrt{|w_{2n+1}^c|}\Delta\hat{X}_{2n+1}(\sqrt{|w_{2n+1}^c|}\Delta\hat{X}_{2n+1})^{\mathrm{T}}$$

$$\sqrt{P_{k/k-1}} = \mathrm{cholupdate}(R, \sqrt{|w_{2n+1}^c|}\Delta\hat{X}_{2n+1}, '-')^{\mathrm{T}} \qquad \mathrm{cholupdate}: R^{\mathrm{T}}R - VV^{\mathrm{T}} \to R_1^{\mathrm{T}}R_1$$

$$\sqrt{P_{ZZ,k/k-1}} = \mathrm{cholupdate}(\left[\sqrt{w_1^c}\Delta\hat{Z}_{1:2n} \quad \sqrt{R_k}\right]^{\mathrm{T}} \to R, \sqrt{|w_{2n+1}^c|}\Delta\hat{Z}_{2n+1}, '-')^{\mathrm{T}}$$

$$\sqrt{P_0} = \sqrt{P_{k/k-1}}; \quad \text{for i=1:m} \quad \sqrt{P_i} = \mathrm{cholupdate}(\sqrt{P_{i-1}}^{\mathrm{T}}, (K_k\sqrt{P_{ZZ,k/k-1}})_i, '-')^{\mathrm{T}}; \quad \text{end for} \quad P_k = \sqrt{P_m}$$

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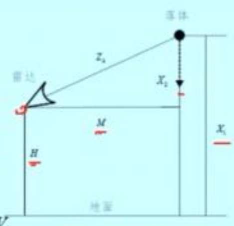
## 14 高斯分布采样非线性滤波

### 14.7 非线性滤波举例

运动学方程 $\begin{cases} \dot{X}_1 = -X_2 \\ \dot{X}_2 = -\dfrac{C\rho A}{2m}X_2^2 \end{cases}$

$f(X)$: $\begin{cases} \dot{X}_1 = -X_2 \\ \dot{X}_2 = -\mathrm{e}^{-\gamma X_1}X_2^2X_3 \\ \dot{X}_3 = 0 \end{cases}$

$h(X_k)$: $Z_k = \sqrt{M^2 + (X_{1,k} - H)^2} + V_k$

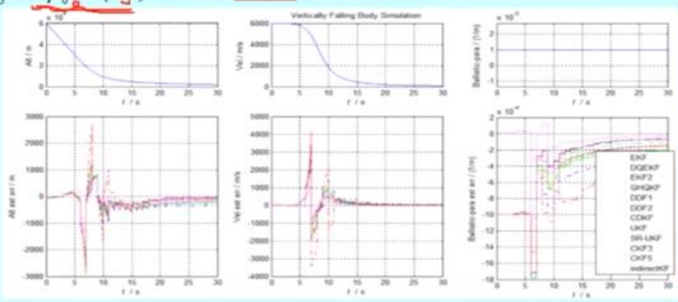


VFB(Vertically Falling Body)

$C, \rho, A, m$ 空气阻力系数、空气密度、迎风面积、落体质量

波尔兹曼公式 $\rho = \rho_0 \mathrm{e}^{-\gamma X_1}$ $\rho_0$ 海平面大气密度，参数 $\gamma \approx 1.7\times10^{-4}(1/\mathrm{m})$

$X_3 = C\rho_0 A/(2m)$ 定义为弹道系数（单位1/m）




$M = H = 30\mathrm{km}$ →初始值，状态方程迭代

$R_k = (100\mathrm{m})^2$

$f_{max} = 1\mathrm{Hz}$

→构造出量测，再滤波估计出状态量

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在高精度惯性组合导航中
非线性滤波方法基本上没用！！！

→惯性用不到的算法比做诈骗


→ 将工程用不到的算法比做诈骗 逆天才

总结：这一节只听个大概思路和思想，具体数学推导一知半解。