姿态解算基础

2023年1月14日 13:14

卡尔曼滤波与组合导航原理(7-9讲)

卡尔曼滤波与组合导航原理
（捷联惯导算法部分）

西北工业大学自动化学院 严恭敏

2020-04

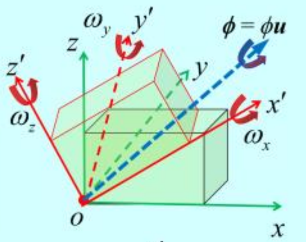
1

---

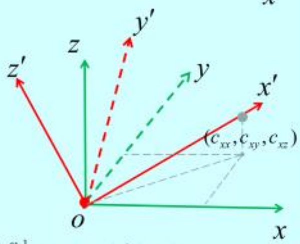
1 姿态解算基础

刚体/坐标系空间定点转动的描述

（1）姿态角(欧拉角) $\boldsymbol{A}=\begin{bmatrix}\theta & \gamma & \psi\end{bmatrix}^{\mathrm{T}}$



（2）姿态阵 $\boldsymbol{C}=\begin{bmatrix}c_{xx} & c_{xy} & c_{xz}\\ c_{yx} & c_{yy} & c_{yz}\\ c_{zx} & c_{zy} & c_{zz}\end{bmatrix}$



（3）等效旋转矢量 $\boldsymbol{\phi}=\begin{bmatrix}\phi_x & \phi_y & \phi_z\end{bmatrix}^{\mathrm{T}}$

（4）四元数 $\boldsymbol{Q}=\cos\frac{\phi}{2}+\frac{\boldsymbol{\phi}}{\phi}\sin\frac{\phi}{2}$

（5）罗德里格参数 $\boldsymbol{\xi}=\boldsymbol{u}\tan\frac{\phi}{2}$

姿态求解目标 $\boldsymbol{A}_0,(\boldsymbol{C}_0/\boldsymbol{\phi}_0/\boldsymbol{Q}_0/\boldsymbol{\xi}_0)\xrightarrow{\boldsymbol{\omega}=[\omega_x,\omega_y,\omega_z]}\boldsymbol{A}_t,(\boldsymbol{C}_t/\boldsymbol{\phi}_t/\boldsymbol{Q}_t/\boldsymbol{\xi}_t)$

初值　陀螺仪测量　实时输出

2

---

1 姿态解算基础

1.1 三阶反对称阵及其矩阵指数函数

设两三维向量

$$\boldsymbol{V}_1=\begin{bmatrix}V_{1x} & V_{1y} & V_{1z}\end{bmatrix}^{\mathrm{T}}$$

$$\boldsymbol{V}_2=\begin{bmatrix}V_{2x} & V_{2y} & V_{2z}\end{bmatrix}^{\mathrm{T}}$$

其叉乘为

$$\boldsymbol{V}_1\times\boldsymbol{V}_2=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ V_{1x} & V_{1y} & V_{1z}\\ V_{2x} & V_{2y} & V_{2z}\end{vmatrix}=\begin{bmatrix}V_{1y}V_{2z}-V_{1z}V_{2y}\\ -(V_{1x}V_{2z}-V_{1z}V_{2x})\\ V_{1x}V_{2y}-V_{1y}V_{2x}\end{bmatrix}$$

若记反对称阵

$$(\boldsymbol{V}\times)=\begin{bmatrix}0 & -V_z & V_y\\ V_z & 0 & -V_x\\ -V_y & V_x & 0\end{bmatrix}$$

则有（改叉乘为矩阵乘）

$$\begin{bmatrix}0 & -V_{1z} & V_{1y}\\ V_{1z} & 0 & -V_{1x}\\ -V_{1y} & V_{1x} & 0\end{bmatrix}\begin{bmatrix}V_{2x}\\ V_{2y}\\ V_{2z}\end{bmatrix}=\begin{bmatrix}V_{1y}V_{2z}-V_{1z}V_{2y}\\ -(V_{1x}V_{2z}-V_{1z}V_{2x})\\ V_{1x}V_{2y}-V_{1y}V_{2x}\end{bmatrix}$$

反对称阵的特性

$$(\boldsymbol{V}\times)^{\mathrm{T}}(\boldsymbol{V}\times)=(\boldsymbol{V}\times)(\boldsymbol{V}\times)^{\mathrm{T}}=\begin{bmatrix}V_y^2+V_z^2 & -V_xV_y & -V_xV_z\\ -V_xV_y & V_x^2+V_z^2 & -V_yV_z\\ -V_xV_z & -V_yV_z & V_x^2+V_y^2\end{bmatrix}$$

反对称阵是正规矩阵(Normal Matrix)，根据矩阵理论可知：正规矩阵可酉相似于对角阵，且不同特征值对应的特征向量两两正交。

3

1 姿态解算基础

反对称阵的特征多项式

$$f(\lambda)=\det[\lambda \boldsymbol{I}-(\boldsymbol{V}\times)]=\begin{vmatrix}\lambda & V_z & -V_y\\ -V_z & \lambda & V_x\\ V_y & -V_x & \lambda\end{vmatrix}$$

$$=\lambda(\lambda^2+V_x^2)-V_z(-\lambda V_z-V_xV_y)-V_y(V_xV_z-\lambda V_y)$$

$$=\lambda^3+(V_x^2+V_y^2+V_z^2)\lambda$$

$$=\lambda(\lambda^2+\upsilon^2) \qquad \upsilon=|\boldsymbol{V}|=\sqrt{V_x^2+V_y^2+V_z^2}$$

求得特征值 $\lambda_1=0,\qquad \lambda_{2,3}=\pm \mathrm{j}\upsilon,$

单位特征向量 $\boldsymbol{u}_1=\dfrac{1}{\upsilon}\begin{bmatrix}V_x\\ V_y\\ V_z\end{bmatrix},\quad \boldsymbol{u}_{2,3}=\dfrac{1}{\upsilon\sqrt{2(V_x^2+V_y^2)}}\begin{bmatrix}-V_xV_z\mp \mathrm{j}\upsilon V_y\\ -V_yV_z\pm \mathrm{j}\upsilon V_x\\ V_x^2+V_y^2\end{bmatrix}$

酉相似于对角阵 $\boldsymbol{\Lambda}=\boldsymbol{U}^{-1}(\boldsymbol{V}\times)\boldsymbol{U}$ $\qquad \boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1\quad \lambda_2\quad \lambda_3)$

$$\boldsymbol{U}=[\boldsymbol{u}_1\quad \boldsymbol{u}_2\quad \boldsymbol{u}_3]$$

1 姿态解算基础

$$(\boldsymbol{V}\times)^2=\begin{bmatrix}0 & -V_z & V_y\\ V_z & 0 & -V_x\\ -V_y & V_x & 0\end{bmatrix}^2=\begin{bmatrix}-V_y^2-V_z^2 & V_xV_y & V_xV_z\\ V_xV_y & -V_x^2-V_z^2 & V_yV_z\\ V_xV_z & V_yV_z & -V_x^2-V_y^2\end{bmatrix}$$

$$\boldsymbol{V}\boldsymbol{V}^{\mathrm{T}}=\begin{bmatrix}V_x\\ V_y\\ V_z\end{bmatrix}\begin{bmatrix}V_x & V_y & V_z\end{bmatrix}=\begin{bmatrix}V_x^2 & V_xV_y & V_xV_z\\ V_xV_y & V_y^2 & V_yV_z\\ V_xV_z & V_yV_z & V_z^2\end{bmatrix}$$

反对称阵的幂方公式

$$(\boldsymbol{V}\times)^1=\upsilon^0(\boldsymbol{V}\times)$$

模

$$(\boldsymbol{V}\times)^2=\boldsymbol{V}\boldsymbol{V}^{\mathrm{T}}-\upsilon^2\boldsymbol{I}=\upsilon^0(\boldsymbol{V}\times)^2$$

$$(\boldsymbol{V}\times)^3=(\boldsymbol{V}\times)^2(\boldsymbol{V}\times)=(\boldsymbol{V}\boldsymbol{V}^{\mathrm{T}}-\upsilon^2\boldsymbol{I})(\boldsymbol{V}\times)=\boldsymbol{V}\boldsymbol{V}^{\mathrm{T}}(\boldsymbol{V}\times)-\upsilon^2(\boldsymbol{V}\times)=\boldsymbol{V}\cdot\boldsymbol{0}_{1\times3}-\upsilon^2(\boldsymbol{V}\times)=-\upsilon^2(\boldsymbol{V}\times)$$

$$(\boldsymbol{V}\times)^4=(\boldsymbol{V}\times)^3(\boldsymbol{V}\times)=-\upsilon^2(\boldsymbol{V}\times)^2$$

$$(\boldsymbol{V}\times)^5=(\boldsymbol{V}\times)^2(\boldsymbol{V}\times)^3=(\boldsymbol{V}\boldsymbol{V}^{\mathrm{T}}-\upsilon^2\boldsymbol{I})\left[-\upsilon^2(\boldsymbol{V}\times)\right]=\upsilon^4(\boldsymbol{V}\times)$$

$$(\boldsymbol{V}\times)^6=(\boldsymbol{V}\times)^3(\boldsymbol{V}\times)^3=\left[-\upsilon^2(\boldsymbol{V}\times)\right]\left[-\upsilon^2(\boldsymbol{V}\times)\right]=\upsilon^4(\boldsymbol{V}\times)^2$$

总结，得幂方通式

$$(\boldsymbol{V}\times)^i=\begin{cases}(-1)^{(i-1)/2}\upsilon^{i-1}(\boldsymbol{V}\times) & i=1,3,5,\cdots\\ (-1)^{(i-2)/2}\upsilon^{i-2}(\boldsymbol{V}\times)^2 & i=2,4,6,\cdots\end{cases}$$

1 姿态解算基础

Hamilton-Cayley定理:方阵满足其零化特征多项式

$$f(\boldsymbol{A})=\boldsymbol{A}^n+a_1\boldsymbol{A}^{n-1}+\cdots a_n\boldsymbol{I}=\boldsymbol{0}\quad\Rightarrow \boldsymbol{A}^m=b_1\boldsymbol{A}^{n-1}+\cdots b_n\boldsymbol{I},\quad m\ge 0$$

反对称阵的矩阵指数函数(推导方法1)

类似泰勒展开

$$\mathrm{e}^{(\boldsymbol{V}\times)}=\sum_{i=0}^{\infty}\frac{(\boldsymbol{V}\times)^i}{i!}=k_0\boldsymbol{I}+k_1(\boldsymbol{V}\times)+k_2(\boldsymbol{V}\times)^2=\underline{\boldsymbol{I}+\frac{\sin\upsilon}{\upsilon}(\boldsymbol{V}\times)+\frac{1-\cos\upsilon}{\upsilon^2}(\boldsymbol{V}\times)^2}$$

为了求待定系数 k_i，利用酉相似特性，有

$$\mathrm{e}^{\boldsymbol{\Lambda}}=\mathrm{e}^{\boldsymbol{U}^{-1}(\boldsymbol{V}\times)\boldsymbol{U}}=\sum_{i=0}^{\infty}\frac{[\boldsymbol{U}^{-1}(\boldsymbol{V}\times)\boldsymbol{U}]^i}{i!}=\boldsymbol{U}^{-1}\left[\sum_{i=0}^{\infty}\frac{(\boldsymbol{V}\times)^i}{i!}\right]\boldsymbol{U}$$

$$=\boldsymbol{U}^{-1}\mathrm{e}^{(\boldsymbol{V}\times)}\boldsymbol{U}=\boldsymbol{U}^{-1}\left[k_0\boldsymbol{I}+k_1(\boldsymbol{V}\times)+k_2(\boldsymbol{V}\times)^2\right]\boldsymbol{U}=k_0\boldsymbol{I}+k_1\boldsymbol{\Lambda}+k_2\boldsymbol{\Lambda}^2$$

即 $\begin{bmatrix}\mathrm{e}^{\lambda_1} & & \\ & \mathrm{e}^{\lambda_2} & \\ & & \mathrm{e}^{\lambda_3}\end{bmatrix}=\begin{bmatrix}k_0+k_1\lambda_1+k_2\lambda_1^2 & & \\ & k_0+k_1\lambda_2+k_2\lambda_2^2 & \\ & & k_0+k_1\lambda_3+k_2\lambda_3^2\end{bmatrix}$

得 $\begin{cases}k_0=1\\ k_0+k_1(\mathrm{j}\upsilon)+k_2(\mathrm{j}\upsilon)^2=\mathrm{e}^{\mathrm{j}\upsilon}=\cos\upsilon+\mathrm{j}\sin\upsilon\\ k_0+k_1(-\mathrm{j}\upsilon)+k_2(-\mathrm{j}\upsilon)^2=\mathrm{e}^{-\mathrm{j}\upsilon}=\cos\upsilon-\mathrm{j}\sin\upsilon\end{cases}\Rightarrow\begin{cases}k_0=1\\ k_1=\dfrac{\sin\upsilon}{\upsilon},\quad k_2=\dfrac{1-\cos\upsilon}{\upsilon^2}\end{cases}$

1 姿态解算基础

反对称阵的矩阵指数函数(推导方法2)

$$\mathrm{e}^{(V\times)}=\sum_{i=0}^{\infty}\frac{(V\times)^i}{i!}=(V\times)^0+\frac{1}{1!}(V\times)^1+\frac{1}{2!}(V\times)^2+\frac{1}{3!}(V\times)^3+\frac{1}{4!}(V\times)^4+\cdots$$

$$=(V\times)^0+\underbrace{\left[\frac{1}{1!}(V\times)^1+\frac{1}{3!}(V\times)^3+\frac{1}{5!}(V\times)^5+\cdots\right]}_{奇}+\underbrace{\left[\frac{1}{2!}(V\times)^2+\frac{1}{4!}(V\times)^4+\frac{1}{6!}(V\times)^6+\cdots\right]}_{偶}$$

利用通式

$$=(V\times)^0+\left[\frac{1}{1!}(V\times)-\frac{\upsilon^2}{3!}(V\times)+\frac{\upsilon^4}{5!}(V\times)+\cdots\right]+\left[\frac{1}{2!}(V\times)^2-\frac{\upsilon^2}{4!}(V\times)^2+\frac{\upsilon^4}{6!}(V\times)^2+\cdots\right]$$

$$=I+\frac{\sin\upsilon}{\upsilon}(V\times)+\frac{1-\cos\upsilon}{\upsilon^2}(V\times)^2$$

矩阵指数函数的特征值与特征向量

$$\mathrm{e}^{(V\times)}U=UU^{-1}\mathrm{e}^{(V\times)}U=U\,\mathrm{e}^{\Lambda}=\begin{bmatrix}\mathrm{e}^{\lambda_1}u_1 & \mathrm{e}^{\lambda_2}u_2 & \mathrm{e}^{\lambda_3}u_3\end{bmatrix}=\begin{bmatrix}1\cdot\frac{V}{\upsilon} & \mathrm{e}^{\lambda_2}u_2 & \mathrm{e}^{\lambda_3}u_3\end{bmatrix}$$

单位正交性 $\left[\mathrm{e}^{(V\times)}\right]^{\mathrm{T}}\mathrm{e}^{(V\times)}=\left[I+\frac{\sin\upsilon}{\upsilon}(V\times)+\frac{1-\cos\upsilon}{\upsilon^2}(V\times)^2\right]^{\mathrm{T}}\cdot\mathrm{e}^{(V\times)}$

$e^{(V\times)}$ 为单位正交阵 $\quad=\left[I+\frac{\sin\upsilon}{\upsilon}(-V\times)^{\mathrm{T}}+\frac{1-\cos\upsilon}{\upsilon^2}(-V\times)^2\right]\mathrm{e}^{(V\times)}=\mathrm{e}^{(-V\times)}\mathrm{e}^{(V\times)}=I$

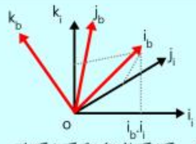
7

以上的纯数学推导会在物理上体现出等效旋转矢量

1 姿态解算基础

1.2 方向余弦阵与等效旋转矢量

两直角坐标系轴上单位矢量之间关系



动系b系和参考系i系

$\cos(\angle i_b i_i)$ 方向余弦

$$\begin{cases}i_b=(i_b\cdot i_i)i_i+(i_b\cdot j_i)j_i+(i_b\cdot k_i)k_i\\ j_b=(j_b\cdot i_i)i_i+(j_b\cdot j_i)j_i+(j_b\cdot k_i)k_i\\ k_b=(k_b\cdot i_i)i_i+(k_b\cdot j_i)j_i+(k_b\cdot k_i)k_i\end{cases}\Rightarrow\begin{bmatrix}i_b & j_b & k_b\end{bmatrix}=\begin{bmatrix}i_i & j_i & k_i\end{bmatrix}\begin{bmatrix}i_b\cdot i_i & j_b\cdot i_i & k_b\cdot i_i\\ i_b\cdot j_i & j_b\cdot j_i & k_b\cdot j_i\\ i_b\cdot k_i & j_b\cdot k_i & k_b\cdot k_i\end{bmatrix}$$

$$=\begin{bmatrix}i_i & j_i & k_i\end{bmatrix}P$$

矢量的坐标表示

$$V=V_x^i i_i+V_y^i j_i+V_z^i k_i=V_x^b i_b+V_y^b j_b+V_z^b k_b$$

P 称为 过渡矩阵 或 坐标系变换矩阵(i->b)

即 $\begin{bmatrix}i_i & j_i & k_i\end{bmatrix}\begin{bmatrix}V_x^i\\V_y^i\\V_z^i\end{bmatrix}=\begin{bmatrix}i_b & j_b & k_b\end{bmatrix}\begin{bmatrix}V_x^b\\V_y^b\\V_z^b\end{bmatrix}=\begin{bmatrix}i_i & j_i & k_i\end{bmatrix}P\begin{bmatrix}V_x^b\\V_y^b\\V_z^b\end{bmatrix}$

有 $\begin{bmatrix}V_x^i\\V_y^i\\V_z^i\end{bmatrix}=P\begin{bmatrix}V_x^b\\V_y^b\\V_z^b\end{bmatrix}$ 记为 $V^i=PV^b=C_b^iV^b$

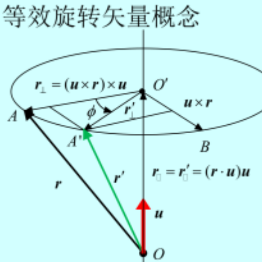
C_b^i 称为 坐标系变换矩阵(i->b)
或 坐标变换矩阵(b->i)

⇒ 姿态阵(方向余弦阵)

8

1 姿态解算基础

等效旋转矢量概念



$r'=f(r,u,\phi)$?

三重矢积公式

$$V_1\times(V_2\times V_3)=(V_1\cdot V_3)V_2-(V_1\cdot V_2)V_3$$

$$(V\cdot V_3)V=V\times(V\times V_3)+\upsilon^2V_3$$

$$=\left[(V\times)^2+\upsilon^2I\right]V_3$$

一些几何关系：

平行的部分(与u平行)

$r=\overline{OO'}+\overline{O'A}\qquad r=r_{/\!/}+r_\perp$ → 与u垂直

$r_{/\!/}=(u\cdot r)u\qquad r_\perp=\overline{O'B}\times u=(u\times r)\times u$

$r'=\overline{OO'}+\overline{O'A'}\qquad r'=r'_{/\!/}+r'_\perp$

$r'_{/\!/}=r_{/\!/}\qquad r'_\perp=\overline{O'A}\cos\phi+\overline{O'B}\sin\phi$

$$=(u\times r)\times u\cos\phi+u\times r\sin\phi$$

因此有

$$r'=(u\cdot r)u+(u\times r)\times u\cos\phi+u\times r\sin\phi$$

$$=\left[I+(u\times)^2\right]r-(u\times)^2r\cos\phi+u\times r\sin\phi$$

$$=\left[I+\sin\phi(u\times)+(1-\cos\phi)(u\times)^2\right]r=Dr$$

⇒ 一个矢量绕另一个矢量旋转的关系式

称为罗德里格旋转公式（Rodrigues） 9

与转轴 u 和转角 φ 有关

1 姿态解算基础

罗德里格旋转公式 $\boldsymbol{r}'=[\boldsymbol{I}+\sin\phi(\boldsymbol{u}\times)+(1-\cos\phi)(\boldsymbol{u}\times)^2]\boldsymbol{r}=\boldsymbol{D}\boldsymbol{r}$

将其应用于直角坐标系的单位坐标轴，有

相当于坐标系三轴的矢量旋转

$$\left.\begin{aligned}\boldsymbol{i}_b&=\boldsymbol{D}\boldsymbol{i}_i\\ \boldsymbol{j}_b&=\boldsymbol{D}\boldsymbol{j}_i\\ \boldsymbol{k}_b&=\boldsymbol{D}\boldsymbol{k}_i\end{aligned}\right\}\longrightarrow \begin{aligned}&[\boldsymbol{i}_b\ \ \boldsymbol{j}_b\ \ \boldsymbol{k}_b]=\boldsymbol{D}[\boldsymbol{i}_i\ \ \boldsymbol{j}_i\ \ \boldsymbol{k}_i] &&\text{基旋转变换关系}\\ &[\boldsymbol{i}_b\ \ \boldsymbol{j}_b\ \ \boldsymbol{k}_b]=[\boldsymbol{i}_i\ \ \boldsymbol{j}_i\ \ \boldsymbol{k}_i]\boldsymbol{P} &&\text{基投影变换关系}\end{aligned}$$

在i系投影 $\left.\begin{aligned}&[\boldsymbol{i}_b^i\ \ \boldsymbol{j}_b^i\ \ \boldsymbol{k}_b^i]=\boldsymbol{D}[\boldsymbol{i}_i^i\ \ \boldsymbol{j}_i^i\ \ \boldsymbol{k}_i^i]=\boldsymbol{D}\boldsymbol{I}\\ &[\boldsymbol{i}_b^i\ \ \boldsymbol{j}_b^i\ \ \boldsymbol{k}_b^i]=[\boldsymbol{i}_i^i\ \ \boldsymbol{j}_i^i\ \ \boldsymbol{k}_i^i]\boldsymbol{P}=\boldsymbol{I}\boldsymbol{P}\end{aligned}\right\}\ \boldsymbol{D}=\boldsymbol{P}=\boldsymbol{C}_b^i$

因此有 $\boldsymbol{D}=\boldsymbol{P}=\boldsymbol{C}_b^i$

矢量/模

$$=\boldsymbol{I}+\sin\phi(\boldsymbol{u}\times)+(1-\cos\phi)(\boldsymbol{u}\times)^2$$

$$\xrightarrow{\boldsymbol{u}=\boldsymbol{\phi}/\phi}=\boldsymbol{I}+\frac{\sin\phi}{\phi}(\boldsymbol{\phi}\times)+\frac{1-\cos\phi}{\phi^2}(\boldsymbol{\phi}\times)^2$$

$$e^{(\boldsymbol{V}\times)}=\boldsymbol{I}+\frac{\sin\upsilon}{\upsilon}(\boldsymbol{V}\times)+\frac{1-\cos\upsilon}{\upsilon^2}(\boldsymbol{V}\times)^2$$

向量的等效旋转矢量含义

$\boldsymbol{V}=\boldsymbol{\phi}=\phi\boldsymbol{u}$ → 数学上的一个向量可以看做物理中的等效旋转矢量

基/坐标系变换矩阵、坐标变换矩阵、罗德里格旋转矩阵、向量反对称阵的指数函数矩阵，四者相互关系与统一。

1 姿态解算基础

$$\boldsymbol{C}_b^i=\boldsymbol{I}+\sin\phi(\boldsymbol{u}\times)+(1-\cos\phi)(\boldsymbol{u}\times)^2$$

$$=\boldsymbol{I}+\frac{\sin\phi}{\phi}(\boldsymbol{\phi}\times)+\frac{1-\cos\phi}{\phi^2}(\boldsymbol{\phi}\times)^2$$

初等旋转（Givens）矩阵：

$$\boldsymbol{C}_b^i\xrightarrow{\boldsymbol{\phi}=[\alpha\ 0\ 0]^{\mathrm{T}}}\begin{bmatrix}1&0&0\\0&\cos\alpha&-\sin\alpha\\0&\sin\alpha&\cos\alpha\end{bmatrix}$$

$$\xrightarrow{\boldsymbol{\phi}=[0\ \beta\ 0]^{\mathrm{T}}}\begin{bmatrix}\cos\beta&0&\sin\beta\\0&1&0\\-\sin\beta&0&\cos\beta\end{bmatrix}$$

$$\xrightarrow{\boldsymbol{\phi}=[0\ 0\ \gamma]^{\mathrm{T}}}\begin{bmatrix}\cos\gamma&-\sin\gamma&0\\\sin\gamma&\cos\gamma&0\\0&0&1\end{bmatrix}$$

矩阵论中常见习题之解法

例：已知 $V_1=[1\ \ 2\ \ 2]^{\mathrm{T}}$ 和 $V_3=[3\ \ 0\ \ 0]^{\mathrm{T}}$

求C，使得：$CV_1=V_3$ 。

$$C_{12}=\begin{bmatrix}1/\sqrt5&2/\sqrt5&0\\-2/\sqrt5&1/\sqrt5&0\\0&0&1\end{bmatrix}$$ 先绕Z轴转使Y为0

$$C_{12}V_1=V_2=[\sqrt5\ \ 0\ \ 2]^{\mathrm{T}}$$

$$C_{13}=\begin{bmatrix}\sqrt5/3&0&2/3\\0&1&0\\-2/3&0&\sqrt5/3\end{bmatrix}$$ 再绕Y轴转使Z为0

$$C_{13}V_2=V_3=[3\ \ 0\ \ 0]^{\mathrm{T}}$$

$$\Rightarrow C=C_{13}C_{12}=\frac{1}{3\sqrt5}\begin{bmatrix}\sqrt5&2\sqrt5&2\sqrt5\\-6&3&0\\-2&-4&5\end{bmatrix}$$ （方法1）

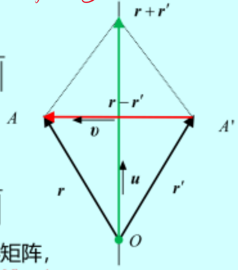
1 姿态解算基础

$$\boldsymbol{C}_b^i=\boldsymbol{I}+\sin\phi(\boldsymbol{u}\times)+(1-\cos\phi)(\boldsymbol{u}\times)^2$$

（方法2）Rodrigues变换： (det(**C**)=1)

$$\boldsymbol{r}'=\boldsymbol{C}_b^i(\pi\boldsymbol{u})\cdot\boldsymbol{r}=\left[\boldsymbol{I}+2(\boldsymbol{u}\times)^2\right]\cdot\boldsymbol{r}\qquad \boldsymbol{u}=\frac{\boldsymbol{r}+\boldsymbol{r}'}{|\boldsymbol{r}+\boldsymbol{r}'|}$$

r→r' 绕对角线转180°



（方法3）Householder变换：

$$\boldsymbol{r}'=\boldsymbol{r}-(\boldsymbol{r}-\boldsymbol{r}')=\boldsymbol{r}-2(\boldsymbol{r}^{\mathrm{T}}\boldsymbol{\upsilon})\boldsymbol{\upsilon}$$

$$=\boldsymbol{r}-2\boldsymbol{\upsilon}\boldsymbol{\upsilon}^{\mathrm{T}}\boldsymbol{r}=(\boldsymbol{I}-2\boldsymbol{\upsilon}\boldsymbol{\upsilon}^{\mathrm{T}})\boldsymbol{r}\qquad \boldsymbol{\upsilon}=\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

$\boldsymbol{H}=\boldsymbol{I}-2\boldsymbol{\upsilon}\boldsymbol{\upsilon}^{\mathrm{T}}$ Householder（反射/镜像）变换矩阵，它为对称的单位正交阵，且det(***H***)=-1。单位正交阵中det(***H***)=-1者都是反射变换矩阵？

```
V1 = [1;2;2]; V3 = [3;0;0];
u = (V1+V3); u = u/norm(u);
C = eye(3)+2*[0,-u(3),u(2);u(3),0,-u(1);-u(2),u(1),0]^2;
v = V1-V3; v = v/norm(v);
H = eye(3)-2*v*v';
[C*V1, H*V1, V3]
```

1 姿态解算基础

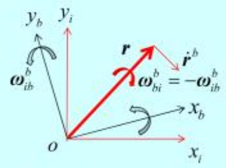
1.3 方向余弦阵微分方程及其毕卡级数解

固定矢量 $\boldsymbol{r}$ 在不同坐标系间的坐标变换关系

$$\boldsymbol{r}^i = \boldsymbol{C}_b^i \boldsymbol{r}^b$$

两边微分，得 $\quad \dot{\boldsymbol{r}}^i = \boldsymbol{C}_b^i \dot{\boldsymbol{r}}^b + \dot{\boldsymbol{C}}_b^i \boldsymbol{r}^b$

$$\xrightarrow[\dot{\boldsymbol{r}}^b = -\boldsymbol{\omega}_{ib}^b \times \boldsymbol{r}^b]{\dot{\boldsymbol{r}}^i = \boldsymbol{0}} \quad \boldsymbol{0} = \boldsymbol{C}_b^i(-\boldsymbol{\omega}_{ib}^b \times \boldsymbol{r}^b) + \dot{\boldsymbol{C}}_b^i \boldsymbol{r}^b$$



固定系(i系)和动系(b系)

整理得 $\quad \dot{\boldsymbol{C}}_b^i \boldsymbol{r}^b = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)\boldsymbol{r}^b$

对于任意不共面三矢量，有 ↓推广

$$\dot{\boldsymbol{C}}_b^i \left[\boldsymbol{r}_1^b \ \ \boldsymbol{r}_2^b \ \ \boldsymbol{r}_3^b\right] = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)\left[\boldsymbol{r}_1^b \ \ \boldsymbol{r}_2^b \ \ \boldsymbol{r}_3^b\right]$$

由此得姿态阵微分方程

$$\dot{\boldsymbol{C}}_b^i = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)$$ →运动学方程

13

b系下陀螺仪输出 + 初值 → 姿态阵就知道了。

1 姿态解算基础

姿态阵微分方程 $\quad \dot{\boldsymbol{C}}_b^i = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)$

由反对称阵的相似变换

先分别投影再运算
=先运算再整体投影

$$\boldsymbol{\omega}_{ib}^i \times \boldsymbol{r}^i = (\boldsymbol{\omega}_{ib} \times \boldsymbol{r})^i = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib} \times \boldsymbol{r})^b$$

$$= \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times \boldsymbol{r}^b) = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)\boldsymbol{r}^b = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)\boldsymbol{C}_i^b \boldsymbol{r}^i$$

$$\Rightarrow (\boldsymbol{\omega}_{ib}^i \times) = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)\boldsymbol{C}_i^b \qquad \text{v.s.} \quad \boldsymbol{\omega}_{ib}^i = \boldsymbol{C}_b^i \boldsymbol{\omega}_{ib}^b$$

第二种推法

因此，有

$$\dot{\boldsymbol{C}}_b^i = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times) \quad = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)\boldsymbol{C}_i^b \boldsymbol{C}_b^i \quad = (\boldsymbol{\omega}_{ib}^i \times)\boldsymbol{C}_b^i$$

同理可获得四种等价的姿态阵微分方程

$$\begin{matrix} \dot{\boldsymbol{C}}_b^i = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times) \\ \dot{\boldsymbol{C}}_b^i = (\boldsymbol{\omega}_{ib}^i \times)\boldsymbol{C}_b^i \end{matrix} \xrightarrow{\text{对换}i,b\text{角标}} \begin{matrix} \dot{\boldsymbol{C}}_i^b = \boldsymbol{C}_i^b(\boldsymbol{\omega}_{bi}^i \times) \\ \dot{\boldsymbol{C}}_i^b = (\boldsymbol{\omega}_{bi}^b \times)\boldsymbol{C}_i^b \end{matrix}$$

14

1 姿态解算基础

简记 $\quad \dot{\boldsymbol{C}}_b^i = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times) \xrightarrow{\boldsymbol{\Omega}(t) = \left[\boldsymbol{\omega}_{ib}^b(t)\times\right]} \dot{\boldsymbol{C}}(t) = \boldsymbol{C}(t)\boldsymbol{\Omega}(t)$ →本质就是什么方程 $\quad$ v.s. $\quad \dot{\boldsymbol{X}}(t) = \boldsymbol{F}(t)\boldsymbol{X}(t)$

姿态阵微分方程的毕卡（Picard）级数解

$$\boldsymbol{C}(t) = \boldsymbol{C}(0) + \int_0^t \boldsymbol{C}(\tau)\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau \qquad [0,t] \text{ 区间积分}$$

$$\boldsymbol{C}(t) = \boldsymbol{C}(0) + \int_0^t \left[\boldsymbol{C}(0) + \int_0^\tau \boldsymbol{C}(\tau_1)\boldsymbol{\Omega}(\tau_1)\,\mathrm{d}\tau_1\right]\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau \qquad \text{代入积分}$$

$$= \boldsymbol{C}(0) + \int_0^t \boldsymbol{C}(0)\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau + \int_0^t\int_0^\tau \boldsymbol{C}(\tau_1)\boldsymbol{\Omega}(\tau_1)\,\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau$$

$$= \boldsymbol{C}(0)\left[\boldsymbol{I} + \int_0^t \boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau\right] + \int_0^t\int_0^\tau \boldsymbol{C}(\tau_1)\boldsymbol{\Omega}(\tau_1)\,\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau$$

$$\boldsymbol{C}(t) = \boldsymbol{C}(0)\left[\boldsymbol{I} + \int_0^t \boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau + \int_0^t\int_0^\tau \boldsymbol{\Omega}(\tau_1)\,\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau\right] + \int_0^t\int_0^\tau\int_0^{\tau_1} \boldsymbol{C}(\tau_2)\boldsymbol{\Omega}(\tau_2)\,\mathrm{d}\tau_2\boldsymbol{\Omega}(\tau_1)\,\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau$$

$$\boldsymbol{C}(t) = \boldsymbol{C}(0)\left[\boldsymbol{I} + \int_0^t \boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau + \int_0^t\int_0^\tau \boldsymbol{\Omega}(\tau_1)\,\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau + \int_0^t\int_0^\tau\int_0^{\tau_1} \boldsymbol{\Omega}(\tau_2)\,\mathrm{d}\tau_2\boldsymbol{\Omega}(\tau_1)\,\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\,\mathrm{d}\tau + \cdots\right]$$

一般情况下，上式即为最终毕卡级数解，不能再进一步化简。15

1 姿态解算基础

矩阵可交换

$$\boldsymbol{C}(t)=\boldsymbol{C}(0)\left[\boldsymbol{I}+\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau+\int_0^t\int_0^\tau\boldsymbol{\Omega}(\tau_1)\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\mathrm{d}\tau+\int_0^t\int_0^\tau\int_0^{\tau_1}\boldsymbol{\Omega}(\tau_2)\mathrm{d}\tau_2\boldsymbol{\Omega}(\tau_1)\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\mathrm{d}\tau+\cdots\right]$$

若转动角速度满足可交换性条件 $\boldsymbol{\Omega}(t)\boldsymbol{\Omega}(\tau)=\boldsymbol{\Omega}(\tau)\boldsymbol{\Omega}(t)$

则有 $\int_0^t\boldsymbol{\Omega}(t)\boldsymbol{\Omega}(\tau)\mathrm{d}\tau=\int_0^t\boldsymbol{\Omega}(\tau)\boldsymbol{\Omega}(t)\mathrm{d}\tau$ 即 $\boldsymbol{\Omega}(t)\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau=\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\boldsymbol{\Omega}(t)$

从而有 $\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\right]^2=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\right]\cdot\left[\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\right]\right\}$

$$=\boldsymbol{\Omega}(t)\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau+\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\boldsymbol{\Omega}(t)=2\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\boldsymbol{\Omega}(t)$$

上式积分可得 $\int_0^t\int_0^\tau\boldsymbol{\Omega}(\tau_1)\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\mathrm{d}\tau=\frac{1}{2}\left[\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\right]^2$

同理可得 $\int_0^t\int_0^{\tau_2}\int_0^{\tau_1}\boldsymbol{\Omega}(\tau_2)\mathrm{d}\tau_2\boldsymbol{\Omega}(\tau_1)\mathrm{d}\tau_1\boldsymbol{\Omega}(\tau)\mathrm{d}\tau=\frac{1}{6}\left[\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\right]^3$ 等等...

综上，得闭合解

$$\boldsymbol{C}(t)=\boldsymbol{C}(0)\left\{\boldsymbol{I}+\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau+\frac{1}{2!}\left[\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\right]^2+\frac{1}{3!}\left[\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau\right]^3+\cdots\right\}=\boldsymbol{C}(0)\mathrm{e}^{\int_0^t\boldsymbol{\Omega}(\tau)\mathrm{d}\tau}$$

16

1 姿态解算基础

可交换性条件 $\boldsymbol{\Omega}(t)\boldsymbol{\Omega}(\tau)=\boldsymbol{\Omega}(\tau)\boldsymbol{\Omega}(t)$ 的含义？

展开成分量形式，有 (记 $t\to1,\tau\to2$)

$$\boldsymbol{\Omega}(t)\boldsymbol{\Omega}(\tau)=[\boldsymbol{\omega}(t)\times][\boldsymbol{\omega}(\tau)\times]=\begin{bmatrix}-\omega_{1y}\omega_{2y}-\omega_{1z}\omega_{2z} & \omega_{1y}\omega_{2x} & \omega_{1z}\omega_{2x}\\ \omega_{1x}\omega_{2y} & -\omega_{1x}\omega_{2x}-\omega_{1z}\omega_{2z} & \omega_{1z}\omega_{2y}\\ \omega_{1x}\omega_{2z} & \omega_{1y}\omega_{2z} & -\omega_{1x}\omega_{2x}-\omega_{1y}\omega_{2y}\end{bmatrix}$$

$$\boldsymbol{\Omega}(\tau)\boldsymbol{\Omega}(t)=[\boldsymbol{\omega}(\tau)\times][\boldsymbol{\omega}(t)\times]=\begin{bmatrix}-\omega_{1y}\omega_{2y}-\omega_{1z}\omega_{2z} & \omega_{2y}\omega_{1x} & \omega_{2z}\omega_{1x}\\ \omega_{2x}\omega_{1y} & -\omega_{1x}\omega_{2x}-\omega_{1z}\omega_{2z} & \omega_{2z}\omega_{1y}\\ \omega_{2x}\omega_{1z} & \omega_{2y}\omega_{1z} & -\omega_{1x}\omega_{2x}-\omega_{1y}\omega_{2y}\end{bmatrix}$$

比较各元素，可得

t时刻和τ时刻的ω成比例

$$\begin{cases}\omega_{1x}\omega_{2y}=\omega_{2x}\omega_{1y}\\ \omega_{1x}\omega_{2z}=\omega_{2x}\omega_{1z}\\ \omega_{1y}\omega_{2z}=\omega_{2y}\omega_{1z}\end{cases}\Longrightarrow\begin{matrix}t\\ \tau\end{matrix}\ \frac{\omega_{1x}}{\omega_{2x}}=\frac{\omega_{1y}}{\omega_{2y}}=\frac{\omega_{1z}}{\omega_{2z}}$$

这表示定轴转动或静止！

17

1 姿态解算基础

定轴转动条件下的姿态阵更新算法

在 $\boldsymbol{C}(T)=\boldsymbol{C}(0)\mathrm{e}^{\int_0^T\boldsymbol{\Omega}(\tau)\mathrm{d}\tau}$ 中

$$\mathrm{e}^{\int_0^T\boldsymbol{\Omega}(\tau)\mathrm{d}\tau}=\mathrm{e}^{\int_0^T[\boldsymbol{\omega}(\tau)\times]\mathrm{d}\tau}=\mathrm{e}^{[\boldsymbol{\theta}(T)\times]}$$

→角度变化矩阵

$$=\boldsymbol{I}+\frac{\sin\theta(T)}{\theta(T)}[\boldsymbol{\theta}(T)\times]+\frac{1-\cos\theta(T)}{\theta^2(T)}[\boldsymbol{\theta}(T)\times]^2$$

开始递推

将时间区间更改为[t_{m-1},t_m]，则有： 例如[0~100s] → 0→1, 1→2, 2→3 ……

$$\boldsymbol{C}_{b(m)}^{i}=\boldsymbol{C}_{b(m-1)}^{i}\boldsymbol{C}_{b(m)}^{b(m-1)}$$

$$\boldsymbol{C}_{b(m)}^{b(m-1)}=\boldsymbol{I}+\frac{\sin\Delta\theta_m}{\Delta\theta_m}(\Delta\boldsymbol{\theta}_m\times)+\frac{1-\cos\Delta\theta_m}{\Delta\theta_m^2}(\Delta\boldsymbol{\theta}_m\times)^2$$

角增量 $\Delta\boldsymbol{\theta}_m=\int_{t_{m-1}}^{t_m}\boldsymbol{\omega}_{ib}^b(t)\mathrm{d}t$

注意:在[t_{m-1},t_m]上须为定轴转动才严格成立！

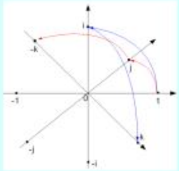
18

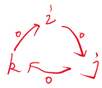
1 姿态解算基础

1.4 姿态更新的四元数描述

$$Q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + \boldsymbol{q}_v$$

×	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1





虚数单位乘法规则 $\begin{cases} i\circ i = j\circ j = k\circ k = -1 \\ i\circ j = k,\quad j\circ k = i,\quad k\circ i = j,\quad j\circ i = -k,\quad k\circ j = -i,\quad i\circ k = -j \end{cases}$

http://math

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$ → 满足结合律
& cut it on a stone of this bridge

Brougham (Broom) Bridge, Dublin

19

1 姿态解算基础

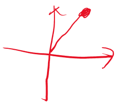
四元数可视为复数概念的扩充 复数的本质是复数

$$Q = (a+bi)+(c+di)j \;\; = a+bi+cj+dij \;\; = a+bi+cj+dk$$

理解 → 复数 表示平面点的位置

类比

四元数表示空间位置



八元数(双/对偶四元数)

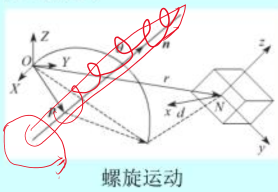
$$\hat{Q} = Q_1 + Q_2 l \quad = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k + \hat{q}_4 l + \hat{q}_5 m + \hat{q}_6 n + \hat{q}_7 o$$

运算规则

$$\begin{aligned}
i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1 \\
i = jk = lm = on = -kj = -ml = -no \\
j = ki = ln = mo = -ik = -nl = -om \\
k = ij = lo = nm = -ji = -ol = -mn \\
l = mi = nj = ok = -im = -jn = -ko \\
m = il = oj = kn = -li = -jo = -nk \\
n = jl = io = mk = -lj = -oi = -km \\
o = ni = jm = kl = -in = -mj = -lk
\end{aligned}$$

不满足交换律和结合律！

物理含义



螺旋运动

十六元数、三十二元数等超复数……

20

二元数 → 复数

1 姿态解算基础

（1）四元数的运算

记
$$P = p_0 + \boldsymbol{p}_v = p_0 + p_1 i + p_2 j + p_3 k$$
$$Q = q_0 + \boldsymbol{q}_v = q_0 + q_1 i + q_2 j + q_3 k$$
$$S = s_0 + \boldsymbol{s}_v = s_0 + s_1 i + s_2 j + s_3 k$$

加减法
$$P \pm Q = (p_0 + p_1 i + p_2 j + p_3 k) \pm (q_0 + q_1 i + q_2 j + q_3 k)$$
$$= (p_0 \pm q_0) + (p_1 \pm q_1) i + (p_2 \pm q_2) j + (p_3 \pm q_3) k$$
$$P \pm Q = (p_0 + \boldsymbol{p}_v) \pm (q_0 + \boldsymbol{q}_v) = (p_0 \pm q_0) + (\boldsymbol{p}_v \pm \boldsymbol{q}_v)$$

乘法
$$P \circ Q = (p_0 + p_1 i + p_2 j + p_3 k) \circ (q_0 + q_1 i + q_2 j + q_3 k)$$
$$= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) + (p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2) i$$
$$+ (p_0 q_2 + p_2 q_0 + p_3 q_1 - p_1 q_3) j + (p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1) k$$
$$P \circ Q = (p_0 + \boldsymbol{p}_v) \circ (q_0 + \boldsymbol{q}_v) = p_0 q_0 + p_0 \boldsymbol{q}_v + q_0 \boldsymbol{p}_v + \boldsymbol{p}_v \circ \boldsymbol{q}_v$$
$$= (p_0 q_0 - \boldsymbol{p}_v^{\mathrm{T}} \boldsymbol{q}_v) + (q_0 \boldsymbol{p}_v + p_0 \boldsymbol{q}_v + \boldsymbol{p}_v \times \boldsymbol{q}_v)$$

→ 所以四元素不满足交换律

21

1 姿态解算基础

$$\boldsymbol{P}\circ\boldsymbol{Q}=(p_0q_0-p_1q_1-p_2q_2-p_3q_3)+(p_0q_1+p_1q_0+p_2q_3-p_3q_2)\boldsymbol{i}+(p_0q_2+p_2q_0+p_2q_1-p_1q_3)\boldsymbol{j}+(p_0q_3+p_3q_0+p_1q_2-p_2q_1)\boldsymbol{k}$$

$$\boldsymbol{P}\circ\boldsymbol{Q}=(p_0q_0-\boldsymbol{p}_v^{\mathrm{T}}\boldsymbol{q}_v)+(q_0\boldsymbol{p}_v+p_0\boldsymbol{q}_v+\boldsymbol{p}_v\times\boldsymbol{q}_v)$$

四元数乘法的矩阵表示

$$\boldsymbol{P}\circ\boldsymbol{Q}=\begin{bmatrix}p_0 & -p_1 & -p_2 & -p_3\\ p_1 & p_0 & -p_3 & p_2\\ p_2 & p_3 & p_0 & -p_1\\ p_3 & -p_2 & p_1 & p_0\end{bmatrix}\begin{bmatrix}q_0\\ q_1\\ q_2\\ q_3\end{bmatrix}\triangleq\boldsymbol{M}_P\boldsymbol{Q}=\begin{bmatrix}q_0 & -q_1 & -q_2 & -q_3\\ q_1 & q_0 & q_3 & -q_2\\ q_2 & -q_3 & q_0 & q_1\\ q_3 & q_2 & -q_1 & q_0\end{bmatrix}\begin{bmatrix}p_0\\ p_1\\ p_2\\ p_3\end{bmatrix}\triangleq\boldsymbol{M}'_Q\boldsymbol{P}$$

$$\boldsymbol{P}\circ\boldsymbol{Q}=\begin{bmatrix}p_0 & -\boldsymbol{p}_v^{\mathrm{T}}\\ \boldsymbol{p}_v & p_0\boldsymbol{I}+(\boldsymbol{p}_v\times)\end{bmatrix}\begin{bmatrix}q_0\\ \boldsymbol{q}_v\end{bmatrix}=\begin{bmatrix}q_0 & -\boldsymbol{q}_v^{\mathrm{T}}\\ \boldsymbol{q}_v & q_0\boldsymbol{I}-(\boldsymbol{q}_v\times)\end{bmatrix}\begin{bmatrix}p_0\\ \boldsymbol{p}_v\end{bmatrix}=\begin{bmatrix}p_0q_0-\boldsymbol{p}_v^{\mathrm{T}}\boldsymbol{q}_v\\ q_0\boldsymbol{p}_v+p_0\boldsymbol{q}_v+\boldsymbol{p}_v\times\boldsymbol{q}_v\end{bmatrix}$$

记三维向量的四维反对称阵

$$(\boldsymbol{p}_v*)_1=\begin{bmatrix}0 & -p_1 & -p_2 & -p_3\\ p_1 & 0 & -p_3 & p_2\\ p_2 & p_3 & 0 & -p_1\\ p_3 & -p_2 & p_1 & 0\end{bmatrix}=\begin{bmatrix}0 & -\boldsymbol{p}_v^{\mathrm{T}}\\ \boldsymbol{p}_v & (\boldsymbol{p}_v\times)\end{bmatrix}\qquad \boldsymbol{M}_P=p_0\boldsymbol{I}+(\boldsymbol{p}_v*)_1$$

$$(\boldsymbol{p}_v*)_2=\begin{bmatrix}0 & -p_1 & -p_2 & -p_3\\ p_1 & 0 & p_3 & -p_2\\ p_2 & -p_3 & 0 & p_1\\ p_3 & p_2 & -p_1 & 0\end{bmatrix}=\begin{bmatrix}0 & -\boldsymbol{p}_v^{\mathrm{T}}\\ \boldsymbol{p}_v & -(\boldsymbol{p}_v\times)\end{bmatrix}\qquad \boldsymbol{M}'_P=p_0\boldsymbol{I}+(\boldsymbol{p}_v*)_2$$

22

1 姿态解算基础

$$(\boldsymbol{P}+\boldsymbol{Q})^*=\boldsymbol{P}^*+\boldsymbol{Q}^*$$
$$(\boldsymbol{P}\circ\boldsymbol{Q})^*=\boldsymbol{Q}^*\circ\boldsymbol{P}^*$$

四元数的共轭 $\boldsymbol{Q}^*=q_0-\boldsymbol{q}_v=q_0-q_1\boldsymbol{i}-q_2\boldsymbol{j}-q_3\boldsymbol{k}$

$$\boldsymbol{Q}\circ\boldsymbol{Q}^*=\boldsymbol{Q}^*\circ\boldsymbol{Q}=q_0^2+\boldsymbol{q}_v^{\mathrm{T}}\boldsymbol{q}_v=q_0^2+q_1^2+q_2^2+q_3^2$$

共轭相乘可交换

四元数的范数 $\|\boldsymbol{Q}\|=\sqrt{\boldsymbol{Q}\circ\boldsymbol{Q}^*}=\sqrt{q_0^2+q_1^2+q_2^2+q_3^2}$

$$\|\boldsymbol{P}\circ\boldsymbol{Q}\|=\|\boldsymbol{Q}\circ\boldsymbol{P}\|=\|\boldsymbol{Q}\|\cdot\|\boldsymbol{P}\|$$

四元数的逆 $\boldsymbol{Q}^{-1}=\dfrac{\boldsymbol{Q}^*}{\|\boldsymbol{Q}\|^2}$

$$\boldsymbol{Q}\circ\boldsymbol{Q}^{-1}=\boldsymbol{Q}^{-1}\circ\boldsymbol{Q}=1$$
$$(\boldsymbol{P}\circ\boldsymbol{Q})^{-1}=\boldsymbol{Q}^{-1}\circ\boldsymbol{P}^{-1}$$

四元数规范化（单位化） $\boldsymbol{Q}=\dfrac{\hat{\boldsymbol{Q}}}{\|\hat{\boldsymbol{Q}}\|^2}$

单位四元数的三角表示 $\boldsymbol{Q}=q_0+\boldsymbol{q}_v=\cos\dfrac{\phi}{2}+\underbrace{\boldsymbol{u}}_{\text{单位矢量}}\sin\dfrac{\phi}{2}$

归一化复数

$\boldsymbol{Z}=\cos\theta+i\sin\theta$

23

1 姿态解算基础

$$\boldsymbol{Q}=q_0+\boldsymbol{q}_v=\cos\frac{\phi}{2}+\boldsymbol{u}\sin\frac{\phi}{2}$$

（2）四元数与姿态阵之间的转换关系

$$\begin{aligned}
\boldsymbol{C}_b^i&=\boldsymbol{I}+\sin\phi(\boldsymbol{u}\times)+(1-\cos\phi)(\boldsymbol{u}\times)^2\\
&=\boldsymbol{I}+2\sin\frac{\phi}{2}\cos\frac{\phi}{2}(\boldsymbol{u}\times)+2\sin^2\frac{\phi}{2}(\boldsymbol{u}\times)^2\\
&=\boldsymbol{I}+2\cos\frac{\phi}{2}(\sin\frac{\phi}{2}\boldsymbol{u}\times)+2(\sin\frac{\phi}{2}\boldsymbol{u}\times)^2\\
&=\boldsymbol{I}+2q_0(\boldsymbol{q}_v\times)+2(\boldsymbol{q}_v\times)^2\\
&=\begin{bmatrix}1-2(q_2^2+q_3^2) & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2)\\ 2(q_1q_2+q_0q_3) & 1-2(q_1^2+q_3^2) & 2(q_2q_3-q_0q_1)\\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & 1-2(q_1^2+q_2^2)\end{bmatrix}\\
&=\begin{bmatrix}q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2)\\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1)\\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2\end{bmatrix}
\end{aligned}$$

倍角公式

$\boldsymbol{Q}\to\boldsymbol{C}_b^i$ √

$\boldsymbol{C}_b^i\to\boldsymbol{Q}$ √

24

1 姿态解算基础

四元数用于坐标变换

左矩阵　右矩阵

$$\boldsymbol{Q}_b^i \circ \boldsymbol{r}^b \circ \boldsymbol{Q}_i^b = \boldsymbol{M}_{Q_b^i}(\boldsymbol{r}^b \circ \boldsymbol{Q}_i^b) = \boldsymbol{M}_{Q_b^i}\left(\boldsymbol{M}'_{Q_i^b}\begin{bmatrix}0\\ \boldsymbol{r}^b\end{bmatrix}\right) = \boldsymbol{M}_{Q_b^i}\boldsymbol{M}'_{Q_i^b}\begin{bmatrix}0\\ \boldsymbol{r}^b\end{bmatrix}$$

→四元素乘法的矩阵表示，见上文

三维矢量认为是0标量四元数

$$= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}\begin{bmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix}\begin{bmatrix}0\\ r_x^b\\ r_y^b\\ r_z^b\end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 0 & 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 0 & 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}\begin{bmatrix}0\\ r_x^b\\ r_y^b\\ r_z^b\end{bmatrix}$$

$$= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C}_b^i \end{bmatrix}\begin{bmatrix}0\\ \boldsymbol{r}^b\end{bmatrix} = \begin{bmatrix}0\\ \boldsymbol{C}_b^i\boldsymbol{r}^b\end{bmatrix} = \begin{bmatrix}0\\ \boldsymbol{r}^i\end{bmatrix}$$

定义成运算符 $\boldsymbol{r}^i = \boldsymbol{Q}_b^i * \boldsymbol{r}^b$ 其等效于坐标变换 $\boldsymbol{r}^i = \boldsymbol{C}_b^i\boldsymbol{r}^b$

25

以上问题说明四元数的作用和矩阵一样

1 姿态解算基础

（3）四元数微分方程

对于固定矢量的四元数坐标变换

$$\boldsymbol{Q}_b^i \circ \boldsymbol{r}^b \circ \boldsymbol{Q}_i^b = \boldsymbol{r}^i \quad \Rightarrow \boldsymbol{Q}_b^i \circ \boldsymbol{r}^b = \boldsymbol{r}^i \circ \boldsymbol{Q}_b^i$$

两边微分得 $\dot{\boldsymbol{Q}}_b^i \circ \boldsymbol{r}^b + \boldsymbol{Q}_b^i \circ \dot{\boldsymbol{r}}^b = \boldsymbol{r}^i \circ \dot{\boldsymbol{Q}}_b^i$

$$\dot{\boldsymbol{Q}}_b^i \circ \boldsymbol{r}^b + \boldsymbol{Q}_b^i \circ (-\boldsymbol{\omega}_{ib}^b \times \boldsymbol{r}^b) = (\boldsymbol{Q}_b^i \circ \boldsymbol{r}^b \circ \boldsymbol{Q}_i^b) \circ \dot{\boldsymbol{Q}}_b^i$$

→?

?

$$\boldsymbol{M}_P = p_0\boldsymbol{I} + (\boldsymbol{p}_v *)_1$$

$$\boldsymbol{M}'_P = p_0\boldsymbol{I} + (\boldsymbol{p}_v *)_2$$

$$\boldsymbol{M}_P - \boldsymbol{M}'_P = \begin{bmatrix} 0 & 0 \\ 0 & 2(\boldsymbol{p}_v \times) \end{bmatrix}$$

$$\boxed{(\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)} \circ \boldsymbol{r}^b - \boldsymbol{r}^b \circ \boxed{(\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)} = \boldsymbol{\omega}_{ib}^b \times \boldsymbol{r}^b$$

左矩阵-右矩阵

$$\underbrace{\left[\boldsymbol{M}_{(\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)} - \boldsymbol{M}'_{(\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)}\right]}_{\begin{bmatrix} 0 & 0 \\ 0 & 2[(\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)_v \times] \end{bmatrix}}\begin{bmatrix}0\\ \boldsymbol{r}^b\end{bmatrix} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_{ib}^b \times \end{bmatrix}\begin{bmatrix}0\\ \boldsymbol{r}^b\end{bmatrix}$$

?

由此得 $(\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)_v = \frac{1}{2}\boldsymbol{\omega}_{ib}^b$　　$(\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)_0 = ?$

26

代表矢量部分　　四元数标量部分

1 姿态解算基础

$$\boldsymbol{Q}_b^i = \begin{bmatrix} \cos\frac{\phi}{2} \\ \boldsymbol{u}_{ib}^b \sin\frac{\phi}{2} \end{bmatrix} \qquad \dot{\boldsymbol{Q}}_b^i = \begin{bmatrix} -\frac{\dot{\phi}}{2}\sin\frac{\phi}{2} \\ \dot{\boldsymbol{u}}_{ib}^b \sin\frac{\phi}{2} + \boldsymbol{u}_{ib}^b \frac{\dot{\phi}}{2}\cos\frac{\phi}{2} \end{bmatrix}$$

四元数及其微分直接相乘，得

$$\boldsymbol{P} \circ \boldsymbol{Q} = \begin{bmatrix} p_0q_0 - \boldsymbol{p}_v^{\mathrm{T}}\boldsymbol{q}_v \\ q_0\boldsymbol{p}_v + p_0\boldsymbol{q}_v + \boldsymbol{p}_v \times \boldsymbol{q}_v \end{bmatrix}$$

$$\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i = \begin{bmatrix} -\frac{\dot{\phi}}{2}\sin\frac{\phi}{2}\cos\frac{\phi}{2} + (\boldsymbol{u}_{ib}^b \sin\frac{\phi}{2})^{\mathrm{T}}(\dot{\boldsymbol{u}}_{ib}^b \sin\frac{\phi}{2} + \boldsymbol{u}_{ib}^b \frac{\dot{\phi}}{2}\cos\frac{\phi}{2}) \\ \cos\frac{\phi}{2}(\dot{\boldsymbol{u}}_{ib}^b \sin\frac{\phi}{2} + \boldsymbol{u}_{ib}^b \frac{\dot{\phi}}{2}\cos\frac{\phi}{2}) + \boldsymbol{u}_{ib}^b \sin\frac{\phi}{2} \cdot \frac{\dot{\phi}}{2}\sin\frac{\phi}{2} - \boldsymbol{u}_{ib}^b \sin\frac{\phi}{2} \times (\dot{\boldsymbol{u}}_{ib}^b \sin\frac{\phi}{2} + \boldsymbol{u}_{ib}^b \frac{\dot{\phi}}{2}\cos\frac{\phi}{2}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \dot{\boldsymbol{u}}_{ib}^b \cos\frac{\phi}{2}\sin\frac{\phi}{2} + \boldsymbol{u}_{ib}^b \frac{\dot{\phi}}{2} - \boldsymbol{u}_{ib}^b \sin\frac{\phi}{2} \times \dot{\boldsymbol{u}}_{ib}^b \sin\frac{\phi}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{u}_{ib}^b\dot{\phi} + \dot{\boldsymbol{u}}_{ib}^b \sin\phi - \boldsymbol{u}_{ib}^b \times \dot{\boldsymbol{u}}_{ib}^b(1-\cos\phi) \end{bmatrix}$$

$$\left.\begin{aligned} (\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)_0 &= 0 \\ (\boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i)_v &= \frac{1}{2}\boldsymbol{\omega}_{ib}^b \end{aligned}\right\} \quad \boldsymbol{Q}_i^b \circ \dot{\boldsymbol{Q}}_b^i = \frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{\omega}_{ib}^b \end{bmatrix}$$

$$\boldsymbol{\omega}_{ib}^b = \boldsymbol{u}_{ib}^b\dot{\phi} + \dot{\boldsymbol{u}}_{ib}^b \sin\phi - \boldsymbol{u}_{ib}^b \times \dot{\boldsymbol{u}}_{ib}^b(1-\cos\phi)$$

等效旋转矢量微分方程的雏形！

至此，得四元数微分方程

$$\dot{\boldsymbol{Q}}_b^i = \frac{1}{2}\boldsymbol{Q}_b^i \circ \boldsymbol{\omega}_{ib}^b$$

27

1 姿态解算基础

$$[\boldsymbol{\omega}(t)\times]_2$$

$$\boldsymbol{\Theta}^2(T)=-\theta^2(T)\boldsymbol{I}$$
$$\boldsymbol{\Theta}^3(T)=\boldsymbol{\Theta}^2(T)\boldsymbol{\Theta}(T)=-\theta^2(T)\boldsymbol{\Theta}(T)$$
$$\boldsymbol{\Theta}^4(T)=\boldsymbol{\Theta}^3(T)\boldsymbol{\Theta}(T)=-\theta^2(T)\boldsymbol{\Theta}(T)\boldsymbol{\Theta}(T)=\theta^4(T)\boldsymbol{I}$$
$$\boldsymbol{\Theta}^5(T)=\boldsymbol{\Theta}^4(T)\boldsymbol{\Theta}(T)=\theta^4(T)\boldsymbol{\Theta}(T)$$
$$\cdots$$

（4）四元数微分方程的求解

$$\dot{\boldsymbol{Q}}_b^i=\frac{1}{2}\boldsymbol{Q}_b^i\circ\boldsymbol{\omega}_{ib}^b\quad\Rightarrow\dot{\boldsymbol{Q}}(t)=\frac{1}{2}\boldsymbol{M}'_{\omega(t)}\boldsymbol{Q}(t)\xrightarrow[{[\boldsymbol{\omega}(t)\times][\boldsymbol{\omega}(\tau)\times]=[\boldsymbol{\omega}(\tau)\times][\boldsymbol{\omega}(t)\times]}]{\text{若具有可交换性}}\boldsymbol{Q}(T)=\mathrm{e}^{\frac{1}{2}\boldsymbol{\Theta}(T)}\boldsymbol{Q}(0)$$

$$\boldsymbol{\Theta}(T)=\int_0^T\boldsymbol{M}'_{\omega(t)}\mathrm{d}t=\int_0^T\begin{bmatrix}0&-\omega_x(T)&-\omega_y(T)&-\omega_z(T)\\\omega_x(T)&0&\omega_z(T)&-\omega_y(T)\\\omega_y(T)&-\omega_z(T)&0&\omega_x(T)\\\omega_z(T)&\omega_y(T)&-\omega_x(T)&0\end{bmatrix}\mathrm{d}t=\begin{bmatrix}0&-\theta_x(T)&-\theta_y(T)&-\theta_z(T)\\\theta_x(T)&0&\theta_z(T)&-\theta_y(T)\\\theta_y(T)&-\theta_z(T)&0&\theta_x(T)\\\theta_z(T)&\theta_y(T)&-\theta_x(T)&0\end{bmatrix}$$

第二类反对称阵

$$\mathrm{e}^{\frac{1}{2}\boldsymbol{\Theta}(T)}=\boldsymbol{I}+\frac{1}{2}\boldsymbol{\Theta}(T)+\frac{[\frac{\boldsymbol{\Theta}(T)}{2}]^2}{2!}+\frac{[\frac{\boldsymbol{\Theta}(T)}{2}]^3}{3!}+\cdots+\frac{[\frac{\boldsymbol{\Theta}(T)}{2}]^n}{n!}+\cdots$$

四阶反对称阵的幂指函数

$$=\boldsymbol{I}+\frac{1}{2}\boldsymbol{\Theta}(T)-\frac{[\frac{\theta(T)}{2}]^2\boldsymbol{I}}{2!}-\frac{[\frac{\theta(T)}{2}]^2\frac{\boldsymbol{\Theta}(T)}{2}}{3!}+\frac{[\frac{\theta(T)}{2}]^4\boldsymbol{I}}{4!}+\frac{[\frac{\theta(T)}{2}]^4\frac{\boldsymbol{\Theta}(T)}{2}}{5!}-\cdots$$

$$=\boldsymbol{I}\{1-\frac{[\frac{\theta(T)}{2}]^2}{2!}+\frac{[\frac{\theta(T)}{2}]^4}{4!}-\cdots\}+\frac{\boldsymbol{\Theta}(T)}{\theta(T)}\{[\frac{\theta(T)}{2}]-\frac{[\frac{\theta(T)}{2}]^3}{3!}+\frac{[\frac{\theta(T)}{2}]^5}{5!}-\cdots\}$$

$$=\boldsymbol{I}\cos\frac{\theta(T)}{2}+\frac{\boldsymbol{\Theta}(T)}{\theta(T)}\sin\frac{\theta(T)}{2}\qquad\text{v.s.}\quad\mathrm{e}^{[\boldsymbol{\theta}(T)\times]}=\boldsymbol{I}+\frac{\sin\theta(T)}{\theta(T)}[\boldsymbol{\theta}(T)\times]+\frac{1-\cos\theta(T)}{\theta^2(T)}[\boldsymbol{\theta}(T)\times]^2$$

1 姿态解算基础

$$\boldsymbol{Q}(T)=\mathrm{e}^{\frac{1}{2}\boldsymbol{\Theta}(T)}\boldsymbol{Q}(0)\qquad\boldsymbol{M}'_P=p_0\boldsymbol{I}+(\boldsymbol{p}_v\times)_2$$

形式一样

$$\boldsymbol{Q}(T)=\underbrace{\left[\boldsymbol{I}\cos\frac{\theta(T)}{2}+\frac{\boldsymbol{\Theta}(T)}{\theta(T)}\sin\frac{\theta(T)}{2}\right]}_{\text{矩阵}}\boldsymbol{Q}(0)=\boldsymbol{Q}(0)\circ\begin{bmatrix}\cos\frac{\theta(T)}{2}\\\frac{\boldsymbol{\theta}(T)}{\theta(T)}\sin\frac{\theta(T)}{2}\end{bmatrix}$$

将时间区间更改为$[t_{m-1},t_m]$，则有四元数更新算法

$$\boldsymbol{Q}_{b(m)}^i=\boldsymbol{Q}_{b(m-1)}^i\circ\boldsymbol{Q}_{b(m)}^{b(m-1)}\qquad\text{v.s.}\quad\begin{cases}\boldsymbol{C}_{b(m)}^i=\boldsymbol{C}_{b(m-1)}^i\boldsymbol{C}_{b(m)}^{b(m-1)}\\\boldsymbol{C}_{b(m)}^{b(m-1)}=\boldsymbol{I}+\frac{\sin\Delta\theta_m}{\Delta\theta_m}(\Delta\boldsymbol{\theta}_m\times)+\frac{1-\cos\Delta\theta_m}{\Delta\theta_m^2}(\Delta\boldsymbol{\theta}_m\times)^2\end{cases}$$

$$\boldsymbol{Q}_{b(m)}^{b(m-1)}=\begin{bmatrix}\cos\frac{\Delta\theta_m}{2}\\\frac{\Delta\boldsymbol{\theta}_m}{\Delta\theta_m}\sin\frac{\Delta\theta_m}{2}\end{bmatrix}\qquad\Delta\boldsymbol{\theta}_m=\int_{t_{m-1}}^{t_m}\boldsymbol{\omega}_{ib}^b\mathrm{d}t\qquad\Delta\theta_m=|\Delta\boldsymbol{\theta}_m|$$

注意:在$[t_{m-1},t_m]$上须为定轴转动才严格成立！

1 姿态解算基础

1.5 等效旋转矢量微分方程 思路.

由 $\boldsymbol{Q}_i^b\circ\dot{\boldsymbol{Q}}_b^i$ （或 $\boldsymbol{C}_i^b\dot{\boldsymbol{C}}_b^i$ ）可推得

$$\boldsymbol{\omega}=\boldsymbol{u}\dot{\phi}+\dot{\boldsymbol{u}}\sin\phi+\dot{\boldsymbol{u}}\times\boldsymbol{u}(1-\cos\phi)$$

将 $\boldsymbol{u}\dot{\phi}=\frac{\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}})}{\phi^2}+\dot{\boldsymbol{\phi}}$, $\dot{\boldsymbol{u}}=-\frac{\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}})}{\phi^3}$, $\dot{\boldsymbol{u}}\times\boldsymbol{u}=\frac{\dot{\boldsymbol{\phi}}\times\boldsymbol{\phi}}{\phi^2}$ (见附录A)代入，得

$$\boldsymbol{\omega}=\left[\frac{\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}})}{\phi^2}+\dot{\boldsymbol{\phi}}\right]-\frac{\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}})}{\phi^3}\sin\phi+\frac{\dot{\boldsymbol{\phi}}\times\boldsymbol{\phi}}{\phi^2}(1-\cos\phi)$$

$$=\dot{\boldsymbol{\phi}}-(1-\cos\phi)\frac{\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}}{\phi^2}+\left(1-\frac{\sin\phi}{\phi}\right)\frac{\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}})}{\phi^2}$$

$$\triangleq\dot{\boldsymbol{\phi}}+a\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}+b(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}}$$

其中记 $a=-\frac{1-\cos\phi}{\phi^2}$, $b=(1-\frac{\sin\phi}{\phi})/\phi^2$

1 姿态解算基础

$$x=g(y,\dot{y})\Rightarrow\dot{y}=f(x,y)$$

$$\boldsymbol{\omega}=\dot{\boldsymbol{\phi}}+a\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}+b(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}}$$

上式左乘 $\boldsymbol{\phi}$：

$$\boldsymbol{\phi}\times\boldsymbol{\omega}=\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}+a(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}}+b(\boldsymbol{\phi}\times)^3\dot{\boldsymbol{\phi}}$$
$$=\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}+a(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}}-b\phi^2\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}$$
$$=(1-b\phi^2)\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}+a(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}}$$

上式左乘$\boldsymbol{\phi}$：$\boldsymbol{\phi}\times\boldsymbol{\omega} = \boldsymbol{\phi}\times\dot{\boldsymbol{\phi}} + a(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}} + b(\boldsymbol{\phi}\times)^3\dot{\boldsymbol{\phi}}$

$$= \boldsymbol{\phi}\times\dot{\boldsymbol{\phi}} + a(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}} - b\phi^2\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}$$

$$= (1-b\phi^2)\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}} + a(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}}$$

再左乘$\boldsymbol{\phi}$：$(\boldsymbol{\phi}\times)^2\boldsymbol{\omega} = (1-b\phi^2)(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}} + a(\boldsymbol{\phi}\times)^3\dot{\boldsymbol{\phi}}$

$$= -a\phi^2\boldsymbol{\phi}\times\dot{\boldsymbol{\phi}} + (1-b\phi^2)(\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}}$$

联立方程组 $\begin{bmatrix}\boldsymbol{\omega}\\ \boldsymbol{\phi}\times\boldsymbol{\omega}\\ (\boldsymbol{\phi}\times)^2\boldsymbol{\omega}\end{bmatrix} = \begin{bmatrix}1 & a & b\\ 0 & 1-b\phi^2 & a\\ 0 & -a\phi^2 & 1-b\phi^2\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{\phi}}\\ \boldsymbol{\phi}\times\dot{\boldsymbol{\phi}}\\ (\boldsymbol{\phi}\times)^2\dot{\boldsymbol{\phi}}\end{bmatrix}$

可求得 $\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} - \dfrac{a}{(1-b\phi^2)^2+a^2\phi^2}\boldsymbol{\phi}\times\boldsymbol{\omega} + \dfrac{a^2-b(1-b\phi^2)}{(1-b\phi^2)^2+a^2\phi^2}(\boldsymbol{\phi}\times)^2\boldsymbol{\omega}$

31

推导思路了解了，但是宏观上完全没理解

1 姿态解算基础

$$a = -\frac{1-\cos\phi}{\phi^2},\quad b = (1-\frac{\sin\phi}{\phi})/\phi^2$$

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} - \frac{a}{(1-b\phi^2)^2+a^2\phi^2}\boldsymbol{\phi}\times\boldsymbol{\omega} + \frac{a^2-b(1-b\phi^2)}{(1-b\phi^2)^2+a^2\phi^2}(\boldsymbol{\phi}\times)^2\boldsymbol{\omega}$$

$$= \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega} + \frac{(1-\cos\phi)^2-(\phi-\sin\phi)\sin\phi}{2\phi^2(1-\cos\phi)}(\boldsymbol{\phi}\times)^2\boldsymbol{\omega}$$

$$= \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega} + \frac{1}{\phi^2}\left(1-\frac{\phi}{2}\cot\frac{\phi}{2}\right)(\boldsymbol{\phi}\times)^2\boldsymbol{\omega} \qquad \text{（Bortz方程，1971年）}$$

当 $\phi \ll 1$ 时，有如下近似：

展开留前两项

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega} + \frac{1}{\phi^2}\left[1-\frac{\phi}{2}\left(\frac{2}{\phi}-\frac{1}{3}\cdot\frac{\phi}{2}-\cancel{\frac{1}{45}\cdot\left(\frac{\phi}{2}\right)^3-\cdots}\right)\right](\boldsymbol{\phi}\times)^2\boldsymbol{\omega}$$

$$\approx \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega} + \frac{1}{12}(\boldsymbol{\phi}\times)^2\boldsymbol{\omega} \qquad \text{（近似1）}$$

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega} \qquad \text{（近似2）}$$

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\omega} \qquad \boldsymbol{\theta} = \int\boldsymbol{\omega}\,\mathrm{d}t \qquad \text{（近似3）}$$

32

1 姿态解算基础

采用广义转动矢量方法推导Bortz方程（更简洁!）

由 $\boldsymbol{Q} = \cos\frac{\phi}{2} + \boldsymbol{u}\sin\frac{\phi}{2}$ 直接微分，得

$$\dot{\boldsymbol{Q}} = -\frac{\dot{\phi}}{2}\sin\frac{\phi}{2} + \left(\dot{\boldsymbol{u}}\sin\frac{\phi}{2} + \boldsymbol{u}\frac{\dot{\phi}}{2}\cos\frac{\phi}{2}\right)$$

另外，由四元数微分方程 $\dot{\boldsymbol{Q}} = \frac{1}{2}\boldsymbol{Q}\circ\boldsymbol{\omega}$ 得

$$\dot{\boldsymbol{Q}} = \frac{1}{2}\left(\cos\frac{\phi}{2} + \boldsymbol{u}\sin\frac{\phi}{2}\right)\circ\boldsymbol{\omega} = -\frac{1}{2}\sin\frac{\phi}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\omega} + \frac{1}{2}\left(\cos\frac{\phi}{2}\boldsymbol{\omega} + \sin\frac{\phi}{2}\boldsymbol{u}\times\boldsymbol{\omega}\right)$$

?

对比上述两式，令标量和矢量分别对应相等，有

$$\begin{cases} -\dfrac{\dot{\phi}}{2}\sin\dfrac{\phi}{2} = -\dfrac{1}{2}\sin\dfrac{\phi}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\omega} \\ \dot{\boldsymbol{u}}\sin\dfrac{\phi}{2} + \boldsymbol{u}\dfrac{\dot{\phi}}{2}\cos\dfrac{\phi}{2} = \dfrac{1}{2}\left(\cos\dfrac{\phi}{2}\boldsymbol{\omega} + \sin\dfrac{\phi}{2}\boldsymbol{u}\times\boldsymbol{\omega}\right) \end{cases} \Rightarrow \begin{cases} \dot{\phi} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\omega} \\ \dot{\boldsymbol{u}} = \dfrac{1}{2}\boldsymbol{u}\times\boldsymbol{\omega} + \dfrac{1}{2}\cot\dfrac{\phi}{2}(\boldsymbol{\omega} - \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\omega}\cdot\boldsymbol{u}) \end{cases}$$

33

1 姿态解算基础

$$\begin{cases} \dot{\phi} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\omega} \\ \dot{\boldsymbol{u}} = \frac{1}{2}\boldsymbol{u}\times\boldsymbol{\omega} + \frac{1}{2}\cot\frac{\phi}{2}\left(\boldsymbol{\omega} - \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\omega}\cdot\boldsymbol{u}\right) \end{cases}$$

$$\boldsymbol{V}_1\times(\boldsymbol{V}_2\times\boldsymbol{V}_3) = (\boldsymbol{V}_1\cdot\boldsymbol{V}_3)\boldsymbol{V}_2 - (\boldsymbol{V}_1\cdot\boldsymbol{V}_2)\boldsymbol{V}_3$$

定义广义转动矢量 $\boldsymbol{g} = f(\phi)\boldsymbol{u}$

将其两边微分，可得

$$\begin{aligned}
\dot{\boldsymbol{g}} &= \frac{\partial f(\phi)}{\partial\phi}\dot{\phi}\boldsymbol{u} + f(\phi)\dot{\boldsymbol{u}} \\
&= \frac{\partial f(\phi)}{\partial\phi}\underline{(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\omega})\boldsymbol{u}} + f(\phi)\left\{\frac{1}{2}\boldsymbol{u}\times\boldsymbol{\omega} + \frac{1}{2}\cot\frac{\phi}{2}\left[\boldsymbol{\omega} - (\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\omega})\boldsymbol{u}\right]\right\} \\
&= \frac{\partial f(\phi)}{\partial\phi}\underline{[\boldsymbol{\omega} + \boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{\omega})]} + f(\phi)\left[\frac{1}{2}\boldsymbol{u}\times\boldsymbol{\omega} - \frac{1}{2}\cot\frac{\phi}{2}\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{\omega})\right] \\
&= \frac{\partial f(\phi)}{\partial\phi}\boldsymbol{\omega} + \frac{1}{2}f(\phi)\boldsymbol{u}\times\boldsymbol{\omega} + \left[\frac{\partial f(\phi)}{\partial\phi} - \frac{1}{2}f(\phi)\cot\frac{\phi}{2}\right]\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{\omega}) \\
&= \frac{\partial f(\phi)}{\partial\phi}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{g}\times\boldsymbol{\omega} + \frac{1}{f^2(\phi)}\left[\frac{\partial f(\phi)}{\partial\phi} - \frac{1}{2}f(\phi)\cot\frac{\phi}{2}\right]\boldsymbol{g}\times(\boldsymbol{g}\times\boldsymbol{\omega})
\end{aligned}$$

若取 $f(\phi) = \phi$，得 $\dot{\boldsymbol{g}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{g}\times\boldsymbol{\omega} + \frac{1}{\phi^2}\left(1 - \frac{\phi}{2}\cot\frac{\phi}{2}\right)\boldsymbol{g}\times(\boldsymbol{g}\times\boldsymbol{\omega})$ 恰好为Bortz方程

34

1 姿态解算基础

$$\dot{\boldsymbol{g}} = \frac{\partial f(\phi)}{\partial\phi}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{g}\times\boldsymbol{\omega} + \frac{1}{f^2(\phi)}\left[\frac{\partial f(\phi)}{\partial\phi} - \frac{1}{2}f(\phi)\cot\frac{\phi}{2}\right]\boldsymbol{g}\times(\boldsymbol{g}\times\boldsymbol{\omega})$$

若取 $f(\phi) = \tan(\phi/2)$，记 $\boldsymbol{\xi} = \tan\frac{\phi}{2}\boldsymbol{u} = \frac{\boldsymbol{u}\sin(\phi/2)}{\cos(\phi/2)} = \frac{\boldsymbol{q}_v}{q_0}$，则有

$$\begin{aligned}
\dot{\boldsymbol{\xi}} &= \frac{1}{2}\sec^2\frac{\phi}{2}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\xi}\times\boldsymbol{\omega} + \frac{1}{\tan^2(\phi/2)}\left(\frac{1}{2}\sec^2\frac{\phi}{2} - \frac{1}{2}\tan\frac{\phi}{2}\cot\frac{\phi}{2}\right)\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{\omega}) \\
&= \frac{1}{2}\left(\tan^2\frac{\phi}{2} + 1\right)\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\xi}\times\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{\omega}) \\
&= \frac{1}{2}\left(\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} + 1\right)\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\xi}\times\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{\omega}) \\
&= \frac{1}{2}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\xi}\times\boldsymbol{\omega} + \frac{1}{2}\left[\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}\boldsymbol{\omega} + \boldsymbol{\xi}\times(\boldsymbol{\xi}\times\boldsymbol{\omega})\right] \\
&= \frac{1}{2}\left[\boldsymbol{I} + (\boldsymbol{\xi}\times) + \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}\right]\boldsymbol{\omega}
\end{aligned}$$

经典罗德里格参数微分方程

35

1 姿态解算基础

$$\dot{\boldsymbol{g}} = \frac{\partial f(\phi)}{\partial\phi}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{g}\times\boldsymbol{\omega} + \frac{1}{f^2(\phi)}\left[\frac{\partial f(\phi)}{\partial\phi} - \frac{1}{2}f(\phi)\cot\frac{\phi}{2}\right]\boldsymbol{g}\times(\boldsymbol{g}\times\boldsymbol{\omega})$$

若取 $f(\phi) = \tan(\phi/4)$，记 $\boldsymbol{\sigma} = \tan\frac{\phi}{4}\boldsymbol{u} = \frac{\boldsymbol{u}\sin(\phi/2)}{1+\cos(\phi/2)} = \frac{\boldsymbol{q}_v}{1+q_0}$，则有

$$\begin{aligned}
\dot{\boldsymbol{\sigma}} &= \frac{1}{4}\sec^2\frac{\phi}{4}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\sigma}\times\boldsymbol{\omega} + \frac{1}{\tan^2(\phi/4)}\left(\frac{1}{4}\sec^2\frac{\phi}{4} - \frac{1}{2}\tan\frac{\phi}{4}\cot\frac{\phi}{2}\right)\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\omega}) \\
&= \frac{1}{4}\left(\tan^2\frac{\phi}{4} + 1\right)\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\sigma}\times\boldsymbol{\omega} + \frac{1}{\tan^2(\phi/4)}\left[\frac{1}{4}\sec^2\frac{\phi}{4} - \frac{1}{4}\left(1 - \tan^2\frac{\phi}{4}\right)\right]\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\omega}) \\
&= \frac{1}{4}(\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma} + 1)\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\sigma}\times\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\omega}) \\
&= \frac{1}{4}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\sigma}\times\boldsymbol{\omega} + \frac{1}{2}\left[\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{\omega} + \boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\omega})\right] - \frac{1}{4}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{\omega} \\
&= \frac{1}{4}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\sigma}\times\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\omega} - \frac{1}{4}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{\omega} \\
&= \frac{1}{4}\left[(1 - \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma})\boldsymbol{I} + 2(\boldsymbol{\sigma}\times) + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\right]\boldsymbol{\omega}
\end{aligned}$$

修正罗德里格参数微分方程

36

广义转动矢量微分方程

Bortz方程
经典罗方程
修正罗方程

1 姿态解算基础

等效旋转矢量微分方程的泰勒级数解法

$$C^i_{b(m)} = C^i_{b(m-1)} C^{b(m-1)}_{b(m)} [\Delta\theta(h)]$$ 定轴转动时才严格成立

$$C^i_{b(m)} = C^i_{b(m-1)} C^{b(m-1)}_{b(m)} [\phi(h)]$$ 非定轴转动时也成立，但关键在于求 $\phi(h)$

当前姿态阵 上一时刻 变化矩阵 等效旋转矢量

$\phi(h)$ 定轴，角增量微分；非定轴，等效旋转矢量微分方程

输入为 ω，输出为 φ(h) 的规律

将 $\phi(h)$ 作泰勒级数展开，得

相当于 $[0,1] \rightarrow [t_{m-1}, t_m]$

$$\phi(h) = \phi(0) + h\dot{\phi}(0) + \frac{h^2}{2!}\ddot{\phi}(0) + \frac{h^3}{3!}\dddot{\phi}(0) + \cdots$$

$$\dot{\phi} = \omega + \frac{1}{2}\theta \times \omega$$

假设角速度为时间线性形式 $\omega(\tau) = a + 2b\tau$

则角增量为 $\Delta\theta(\tau) = \int_0^\tau \omega(\tau_1) d\tau_1 = a\tau + b\tau^2$

且有
$$\begin{cases} \omega(0) = a \\ \dot{\omega}(0) = 2b \\ \omega^{(i)}(0) = 0 \quad i = 2,3,4,\cdots \end{cases} \qquad \begin{cases} \Delta\theta(0) = 0 \\ \Delta\dot{\theta}(0) = \omega(0) = a \\ \Delta\ddot{\theta}(0) = \dot{\omega}(0) = 2b \\ \Delta\theta^{(i)}(0) = \omega^{(i-1)}(0) = 0 \quad i = 3,4,5,\cdots \end{cases}$$

37

1 姿态解算基础

$$\begin{cases} \omega(0) = a \\ \dot{\omega}(0) = 2b \\ \omega^{(i)}(0) = 0 \quad i = 2,3,4,\cdots \end{cases} \qquad \begin{cases} \Delta\theta(0) = 0 \\ \Delta\dot{\theta}(0) = \omega(0) = a \\ \Delta\ddot{\theta}(0) = \dot{\omega}(0) = 2b \\ \Delta\theta^{(i)}(0) = \omega^{(i-1)}(0) = 0 \quad i = 3,4,5,\cdots \end{cases}$$

记 $\beta(\tau) = \Delta\theta(\tau) \times \omega(\tau)$

则有
$$\begin{cases} \beta(0) = 0 \\ \dot{\beta}(0) = C_1^0 \Delta\dot{\theta}(0) \times \omega(t_k) + C_1^1 \cdot 0 = a \times a = 0 \\ \ddot{\beta}(0) = C_2^0 \Delta\ddot{\theta}(0) \times \omega(t_k) + C_2^1 \Delta\dot{\theta}(0) \times \dot{\omega}(t_k) + C_2^2 \cdot 0 = 2b \times a + 2 \cdot a \times 2b = 2a \times b \\ \beta^{(i)}(0) = 0 \quad i = 3,4,5,\cdots \end{cases}$$

根据 $\dot{\phi}(t) = \omega(t) + \frac{1}{2}\theta(t) \times \omega(t) = \omega(t) + \frac{1}{2}\beta(t)$，有

$$\begin{cases} \dot{\phi}(0) = \omega(0) + \frac{1}{2}\beta(0) = \omega(0) = a \\ \ddot{\phi}(0) = \dot{\omega}(0) + \frac{1}{2}\dot{\beta}(0) = \dot{\omega}(0) = 2b \\ \dddot{\phi}(0) = \ddot{\omega}(0) + \frac{1}{2}\ddot{\beta}(0) = \frac{1}{2}\ddot{\beta}(0) = a \times b \\ \phi^{(i)}(0) = 0 \quad i = 4,5,6,\cdots \end{cases}$$

至此得 $\phi(h) = \phi(0) + h\dot{\phi}(0) + \frac{h^2}{2!}\ddot{\phi}(0) + \frac{h^3}{3!}\dddot{\phi}(0) + \cdots = ha + h^2 b + \frac{h^3}{6} a \times b$

38

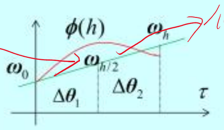
1 姿态解算基础

$$\phi(h) = ha + h^2 b + \frac{h^3}{6} a \times b$$

陀螺仪 + 加振器（数字滤波可能会损失信息，延迟太高）

假设在一次姿态更新周期内进行了两次角增量采样，有

$$\begin{cases} \Delta\theta_1 = \int_0^{h/2} \omega(\tau) d\tau = a\tau + b\tau^2 \Big|_0^{h/2} = \frac{h}{2}a + \frac{h^2}{4}b \\ \Delta\theta_2 = \int_{h/2}^{h} \omega(\tau) d\tau = a\tau + b\tau^2 \Big|_{h/2}^{h} = \frac{h}{2}a + \frac{3h^2}{4}b \end{cases}$$



上述三个方程联立，消去 **a,b** 参数，可求得

$$\phi(h) = \underbrace{(\Delta\theta_1 + \Delta\theta_2)}_{角增量} + \frac{2}{3}\Delta\theta_1 \times \Delta\theta_2$$ （等效旋转矢量二子样算法）

→ 定轴：θ₁ ∥ θ₂ =0

→ 非定轴，θ₁ 不∥ θ₂ ≠0 为修正

同理，可推导得其它子样算法...

$$\phi(h) = (\Delta\theta_1 + \Delta\theta_2 + \Delta\theta_3) + \frac{33}{80}\Delta\theta_1 \times \Delta\theta_3 + \frac{57}{80}(\Delta\theta_1 \times \Delta\theta_2 + \Delta\theta_2 \times \Delta\theta_3)$$ （三子样）

$$\phi(h) = \Delta\theta_1 + \frac{1}{12}\Delta\theta_0 \times \Delta\theta_1$$ （单子样+前一周期）

39

以上是在假设角速度为时间的线性函数基础上，可扩展为多项式运动，推出等效旋转矢量

以下是在圆锥运动基础上

1 姿态解算基础

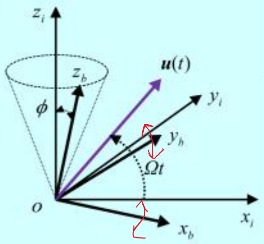
1.5 圆锥运动条件下的等效旋转矢量算法

圆锥运动的角速度描述 $\omega(t) = [a\Omega \sin \Omega t \quad b\Omega \cos \Omega t \quad c]^T$ → 两轴是振动，

→ ω(t) = [aΩ sin Ωt, bΩ sin ωt, 0]ᵀ 即是定轴转动

圆锥运动的角速度描述 $\boldsymbol{\omega}(t)=[a\Omega\sin\Omega t \quad b\Omega\cos\Omega t \quad c]^{T}$

$\rightarrow \omega(t)=[a\Omega\sin\Omega t,\ b\Omega\sin\omega t,\ 0]^T$ 即是定轴转动



z_b轴在做圆锥运动，x_b、y_b轴做角振动（以x_i, y_i为中心）

40

1 姿态解算基础

圆锥运动的四元数描述

$$\boldsymbol{Q}(t)=\begin{bmatrix}\cos(\phi/2)\\ \dfrac{\boldsymbol{\phi}}{\phi}\sin(\phi/2)\end{bmatrix} \Rightarrow \boldsymbol{\phi}(t)=\phi\begin{bmatrix}\cos\Omega t\\ \sin\Omega t\\ 0\end{bmatrix}$$

→ 等效旋转矢量

? 怎么来的

$$\boldsymbol{Q}(t)=\begin{bmatrix}\cos(\phi/2)\\ \sin(\phi/2)\cos\Omega t\\ \sin(\phi/2)\sin\Omega t\\ 0\end{bmatrix}$$

其微分 $$\dot{\boldsymbol{Q}}(t)=\begin{bmatrix}0\\ -\Omega\sin(\phi/2)\sin\Omega t\\ \Omega\sin(\phi/2)\cos\Omega t\\ 0\end{bmatrix}=\Omega\sin\frac{\phi}{2}\cdot\begin{bmatrix}0\\ -\sin\Omega t\\ \cos\Omega t\\ 0\end{bmatrix}$$

由公式 $\dot{\boldsymbol{Q}}(t)=\frac{1}{2}\boldsymbol{Q}(t)\circ\boldsymbol{\omega}_q(t)$ 得角速度表达式

$$\boldsymbol{\omega}_q(t)=2\boldsymbol{Q}^*(t)\circ\dot{\boldsymbol{Q}}(t)=2\boldsymbol{M}'_{\dot{Q}(t)}\boldsymbol{Q}^*(t)$$

$$=2\Omega\sin\frac{\phi}{2}\cdot\begin{bmatrix}0 & \sin\Omega t & -\cos\Omega t & 0\\ -\sin\Omega t & 0 & 0 & -\cos\Omega t\\ \cos\Omega t & 0 & 0 & -\sin\Omega t\\ 0 & \cos\Omega t & \sin\Omega t & 0\end{bmatrix}\begin{bmatrix}\cos(\phi/2)\\ -\sin(\phi/2)\cos\Omega t\\ -\sin(\phi/2)\sin\Omega t\\ 0\end{bmatrix}=\begin{bmatrix}0\\ -\Omega\sin\phi\sin\Omega t\\ \Omega\sin\phi\cos\Omega t\\ -2\Omega\sin^2(\phi/2)\end{bmatrix}$$

与 $\omega(t)=[a\Omega\sin\Omega t,\ b\Omega\cos\Omega t,\ \omega]^T$ 形式相同

$$\Rightarrow\boldsymbol{\omega}(t)=\begin{bmatrix}-\Omega\sin\phi\sin\Omega t\\ \Omega\sin\phi\cos\Omega t\\ -2\Omega\sin^2(\phi/2)\end{bmatrix}=\Omega\sin\phi\begin{bmatrix}-\sin\Omega t\\ \cos\Omega t\\ -\tan(\phi/2)\end{bmatrix}$$

41

1 姿态解算基础

$$\dot{\boldsymbol{\phi}}=\boldsymbol{\omega}+\frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega}+\frac{1}{\phi^2}(1-\frac{\phi}{2}\cot\frac{\phi}{2})\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega})$$

← 代入后验证

$$\boldsymbol{\phi}(t)=\phi\begin{bmatrix}\cos\Omega t\\ \sin\Omega t\\ 0\end{bmatrix} \qquad \boldsymbol{\omega}(t)=\Omega\sin\phi\begin{bmatrix}-\sin\Omega t\\ \cos\Omega t\\ -\tan(\phi/2)\end{bmatrix}$$

除定轴转动外，圆锥运动是满足Bortz方程的一种最简形式运动：

$$\boldsymbol{\phi}\times\boldsymbol{\omega}=\phi\begin{bmatrix}\cos\Omega t\\ \sin\Omega t\\ 0\end{bmatrix}\times\left(\Omega\sin\phi\begin{bmatrix}-\sin\Omega t\\ \cos\Omega t\\ -\tan(\phi/2)\end{bmatrix}\right)=\phi\Omega\sin\phi\begin{bmatrix}-\sin\Omega t\tan(\phi/2)\\ \cos\Omega t\tan(\phi/2)\\ 1\end{bmatrix}$$

$$\boldsymbol{\phi}\times(\boldsymbol{\phi}\times\boldsymbol{\omega})=\phi\begin{bmatrix}\cos\Omega t\\ \sin\Omega t\\ 0\end{bmatrix}\times\left(\phi\Omega\sin\phi\begin{bmatrix}-\sin\Omega t\tan(\phi/2)\\ \cos\Omega t\tan(\phi/2)\\ 1\end{bmatrix}\right)=\phi^2\Omega\sin\phi\begin{bmatrix}\sin\Omega t\\ -\cos\Omega t\\ \tan(\phi/2)\end{bmatrix}=-\phi^2\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\phi}}=\boldsymbol{\omega}+\frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega}+\frac{1}{\phi^2}\left(1-\frac{\phi}{2}\cot\frac{\phi}{2}\right)(-\phi^2\boldsymbol{\omega})=\underline{\frac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega}+\frac{\phi}{2}\cot\frac{\phi}{2}\boldsymbol{\omega}}$$

$$=\frac{1}{2}\phi\Omega\sin\phi\begin{bmatrix}-\sin\Omega t\tan(\phi/2)\\ \cos\Omega t\tan(\phi/2)\\ 1\end{bmatrix}+\frac{\phi}{2}\cot\frac{\phi}{2}\cdot\Omega\sin\phi\begin{bmatrix}-\sin\Omega t\\ \cos\Omega t\\ -\tan(\phi/2)\end{bmatrix}$$

$$=\frac{1}{2}\phi\Omega\begin{bmatrix}-\sin\Omega t\cdot[\tan(\phi/2)+\cot(\phi/2)]\sin\phi\\ \cos\Omega t\cdot[\tan(\phi/2)+\cot(\phi/2)]\sin\phi\\ 0\end{bmatrix}=\phi\Omega\begin{bmatrix}-\sin\Omega t\\ \cos\Omega t\\ 0\end{bmatrix}$$

定轴转动：

$$\dot{\boldsymbol{\phi}}=\boldsymbol{\omega} \qquad \boldsymbol{\omega}(t)=\begin{bmatrix}a\omega(t)\\ b\omega(t)\\ c\omega(t)\end{bmatrix}$$

旋转弹章动...? → 发射的子弹

$$\boldsymbol{\omega}(t)=\begin{bmatrix}-\sin\Omega t\\ \cos\Omega t\\ \omega(t)\end{bmatrix}$$

42

↳ 圆锥运动为章动的一个特例

1 姿态解算基础

$$Q(T)=\cos\frac{\phi(T)}{2}+\frac{\boldsymbol{\phi}(T)}{\phi(T)}\sin\frac{\phi(T)}{2}$$ →四元数与等效旋转矢量的关系

圆锥误差补偿多子样**优化**系数的求解

根据四元数更新方程 $Q(t_m)=Q(t_{m-1})\circ Q(T)$ 求四元数变化/增量

这一刻　上一刻　增量

$$Q(T)=Q^*(t_{m-1})\circ Q(t_m)=M_{Q^*(t_{m-1})}Q(t_m)$$

$$=\begin{bmatrix}\cos\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\Omega t_{m-1} & \sin\frac{\phi}{2}\sin\Omega t_{m-1} & 0\\ -\sin\frac{\phi}{2}\cos\Omega t_{m-1} & \cos\frac{\phi}{2} & 0 & -\sin\frac{\phi}{2}\sin\Omega t_{m-1}\\ -\sin\frac{\phi}{2}\sin\Omega t_{m-1} & 0 & \cos\frac{\phi}{2} & \sin\frac{\phi}{2}\cos\Omega t_{m-1}\\ 0 & \sin\frac{\phi}{2}\sin\Omega t_{m-1} & -\sin\frac{\phi}{2}\cos\Omega t_{m-1} & \cos\frac{\phi}{2}\end{bmatrix}\begin{bmatrix}\cos\frac{\phi}{2}\\ \sin\frac{\phi}{2}\cos\Omega t_m\\ \sin\frac{\phi}{2}\sin\Omega t_m\\ 0\end{bmatrix}=\begin{bmatrix}1-2(\sin\frac{\phi}{2}\cos\frac{\Omega T}{2})^2\\ -\sin\phi\sin\frac{\Omega T}{2}\sin\Omega(t_m-\frac{T}{2})\\ \sin\phi\sin\frac{\Omega T}{2}\cos\Omega(t_m-\frac{T}{2})\\ -\sin^2\frac{\phi}{2}\sin\Omega T\end{bmatrix}$$

再求等效旋转矢量增量(只能是近似！)

sinx的展开

$$\frac{\boldsymbol{\phi}(T)}{\phi(T)}\sin\frac{\phi(T)}{2}=\boldsymbol{q}_v(T)=\begin{bmatrix}-\sin\phi\sin\frac{\Omega T}{2}\sin\Omega(t_m-\frac{T}{2})\\ \sin\phi\sin\frac{\Omega T}{2}\cos\Omega(t_m-\frac{T}{2})\\ -\sin^2\frac{\phi}{2}\sin\Omega T\end{bmatrix}\Rightarrow\boldsymbol{\phi}(T)\approx\begin{bmatrix}-2\sin\phi\sin\frac{\Omega T}{2}\sin\Omega(t_m-\frac{h}{2})\\ 2\sin\phi\sin\frac{\Omega T}{2}\cos\Omega(t_m-\frac{h}{2})\\ -2\sin^2\frac{\phi}{2}\sin\Omega T\end{bmatrix}$$

43

已经推出了圆锥运动的等效旋转矢量，如何再用角增量构造出这些等效旋转矢量

→是角速度测得的角增量

1 姿态解算基础

圆锥运动条件下的角增量

$$\boldsymbol{\phi}(T)\approx\begin{bmatrix}-2\sin\phi\sin\frac{\Omega T}{2}\sin\Omega(t_m-\frac{T}{2})\\ 2\sin\phi\sin\frac{\Omega T}{2}\cos\Omega(t_m-\frac{T}{2})\\ -2\sin^2\frac{\phi}{2}\sin\Omega T\end{bmatrix}$$ →目标量(理论值)

$$\Delta\boldsymbol{\theta}_m=\int_{t_{m-1}}^{t_m}\boldsymbol{\omega}(t)\mathrm{d}t=\int_{t_{m-1}}^{t_m}\begin{bmatrix}-\Omega\sin\phi\sin\Omega t\\ \Omega\sin\phi\cos\Omega t\\ -2\Omega\sin^2(\phi/2)\end{bmatrix}\mathrm{d}t$$

$$=\begin{bmatrix}\sin\phi\cdot(\cos\Omega t_m-\cos\Omega t_{m-1})\\ \sin\phi\cdot(\sin\Omega t_m-\sin\Omega t_{m-1})\\ -2\sin^2(\phi/2)\cdot\Omega T\end{bmatrix}=\begin{bmatrix}-2\sin\phi\sin\frac{\Omega T}{2}\sin\Omega(t_m-\frac{T}{2})\\ 2\sin\phi\sin\frac{\Omega T}{2}\cos\Omega(t_m-\frac{T}{2})\\ -2\sin^2\frac{\phi}{2}\cdot\Omega T\end{bmatrix}$$ →量测量，

→角差别是多少，需要补偿

对比**角增量**与**等效旋转矢量增量**，得差异量(待补偿量)为

$$\delta\boldsymbol{\phi}(T)=\boldsymbol{\phi}(T)-\Delta\boldsymbol{\theta}_m$$

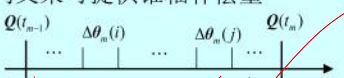
$$=\begin{bmatrix}0\\ 0\\ -2\sin^2\frac{\phi}{2}\sin\Omega T-(-2\sin^2\frac{\phi}{2}\cdot\Omega T)\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 2\sin^2\frac{\phi}{2}(\Omega T-\sin\Omega T)\end{bmatrix}$$

44

1 姿态解算基础

利用子样间叉乘可提供锥轴补偿量

$$\delta\boldsymbol{\phi}(T)=\begin{bmatrix}0\\ 0\\ 2\sin^2\frac{\phi}{2}(\Omega T-\sin\Omega T)\end{bmatrix}$$



Y轴正弦运动，Z轴余弦运动

X乘后，Z轴正好为常值

补偿

$$\Delta\boldsymbol{\theta}_m(i)\times\Delta\boldsymbol{\theta}_m(j)\approx\begin{bmatrix}\frac{(i-j)(\phi\lambda)^3}{2}\sin\Omega(t_{m-1}+\frac{i+j-1}{2}h)\\ -\frac{(i-j)(\phi\lambda)^3}{2}\cos\Omega(t_{m-1}+\frac{i+j-1}{2}h)\\ -4\sin^2\phi\sin^2\frac{\lambda}{2}\sin(i-j)\lambda\end{bmatrix}$$

补偿→只补偿 sinΩT 的低阶项，高阶项称为剩余误差

$\lambda=\Omega T/N$

对比**待补偿量**与**可提供补偿量**，可求得(近似忽略非锥轴影响)

N子样算法 $\delta\hat{\boldsymbol{\phi}}(T)=\sum_{i=1}^{N-1}k_{N-i}\Delta\boldsymbol{\theta}_m(i)\times\Delta\boldsymbol{\theta}_m(N)$

及其剩余误差 $\varepsilon_N=\rho_N\frac{\phi^2(\Omega T)^{2N+1}}{T}$

45

如何求系数 k_{N-i}，ρ_N 编程算吧

1 姿态解算基础

$$\delta\hat{\boldsymbol{\phi}}(T)=\sum_{i=1}^{N-1}k_{N-i}\Delta\boldsymbol{\theta}_m(i)\times\Delta\boldsymbol{\theta}_m(N)\qquad\varepsilon_N=\rho_N\frac{\phi^2(\Omega T)^{2N+1}}{T}$$

N	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	ρ_N
1	-									8.333E-02
2	0.667									1.042E-03
3	0.450	1.350								4.899E-06
4	0.514	0.876	2.038							1.211E-08

→多项式运动条件下一项

→后面的就不同了

有差别

① 一般高精度陀螺 $10^{-3}\sim10^{-4}$ 精度

② 而超高精度陀螺 10^{-8}，也会

N	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	ρ_N
1	-									8.333E-02
2	0.667									1.042E-03
3	0.450	1.350								4.899E-06
4	0.514	0.876	2.038							1.211E-08
5	0.496	1.042	1.290	2.728						1.847E-11
6	0.501	0.987	1.579	1.696	3.419					1.912E-14
7	0.500	1.004	1.471	2.124	2.097	4.111				1.432E-17
8	0.500	0.999	1.510	1.951	2.676	2.495	4.083			8.119E-21
9	0.500	1.000	1.497	2.018	2.426	3.231	2.891	5.495		3.606E-24
10	0.500	1.000	1.501	1.993	2.529	2.898	3.790	3.285	6.178	1.289E-27

→ 多项式运动条件下一项

→ 后面的就不同了

有意义

算法误差越来越小

① 一般高精度陀螺 10 ~10 精度

② 而超高精度陀螺 10^{-8}，也不会在恶劣环境下使用（不会产生圆锥运动）

③ 所以过高精度的算法也不实用

讨论：

(1)推导过程中作了近似（等效旋转矢量增量的小量近似、非锥轴影响近似忽略），当锥角不是很小时并非子样数越高越好，多数文献过度研究；

(2)实际陀螺误差（噪声/分辨率、幅相特性误差、动态误差）远比载体角运动激励引起的圆锥误差和算法更新原理误差大；

(3)算法误差可简单地通过提高采样频率改善，实际系统中使用二子样至多三子样即可，若依然无法解决精度问题那就不是子样数多少的问题了。 46

1 姿态解算基础

姿态更新算法总结：

九参数 $\dot{\boldsymbol{C}}_b^i = \boldsymbol{C}_b^i(\boldsymbol{\omega}_{ib}^b \times)$ 姿态阵

四参数 $\dot{\boldsymbol{Q}}_b^i = \frac{1}{2}\boldsymbol{Q}_b^i \circ \boldsymbol{\omega}_{ib}^b$ 四元数

线性方程

三参数 $\dot{\boldsymbol{\phi}} = \left[\boldsymbol{I} + \frac{1}{2}(\boldsymbol{\phi}\times) + \frac{1}{\phi^2}\left(1 - \frac{\phi}{2}\cot\frac{\phi}{2}\right)(\boldsymbol{\phi}\times)^2\right]\boldsymbol{\omega}$ 等效旋转矢量

$\dot{\boldsymbol{\xi}} = \frac{1}{2}\left[\boldsymbol{I} + (\boldsymbol{\xi}\times) + \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}\right]\boldsymbol{\omega}$ 罗德里格

非线性

$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1 - \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma})\boldsymbol{I} + 2(\boldsymbol{\sigma}\times) + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\right]\boldsymbol{\omega}$ 修正～ $\dot{y} = f(y)x$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\gamma} \\ \dot{\psi} \end{bmatrix} = \frac{1}{\mathrm{c}_\theta}\begin{bmatrix} \mathrm{c}_\gamma \mathrm{c}_\theta & 0 & \mathrm{c}_\theta \mathrm{s}_\gamma \\ \mathrm{s}_\theta \mathrm{s}_\gamma & \mathrm{c}_\theta & -\mathrm{s}_\theta \mathrm{c}_\gamma \\ -\mathrm{s}_\gamma & 0 & \mathrm{c}_\gamma \end{bmatrix}\boldsymbol{\omega}$$ 欧拉角

当前主流算法步骤：$\Delta\boldsymbol{\theta}_{mi} \xrightarrow{\text{多子样}} \boldsymbol{\phi} \longrightarrow \boldsymbol{Q}$ （或者 C） 零误差 47

陀螺采样 等效旋转矢量 四元数/姿态阵

⇒ 得到动力学方程的姿态参数 ⇒ 进行控制

导航 → 控制